1^{st} oct :

Theorm 3.14 : If $\sum a_n$ converges absolutly, then every rearragement of $\sum a_n$ cowg to the same sum. (Note : Two sums may not be equal)

proof : let $\sum b_n$ be a rearragement of $\sum a_n$, with partial sums t_n.

given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t $m \geq n \geq N$

$$\sum_{i=n}^{m} |a_i| \leq \varepsilon \quad \text{by cauchy condition for } \sum |a_n|$$

let $\{k_n\}_{n=1}^{\infty}$ be a sequence of indices with will the terms $\{b_n\}_{n=1}^{\infty}$ occurs, in the $\{a_n\}_{n=1}^{\infty}$ in order.

let $n_0 \in \mathbb{N}$ s.t

$$1, 2, 3, \ldots N \in \{k_1, k_2, \ldots k_{n_0}\}$$

(Note : $\sum a_n$ is s_n , t.s that $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t
$\sum b_n$ is t_n , $|s_n - t_n| \leq \varepsilon \quad \forall n \geq N$)

($b_1 + b_2 + b_3 + \cdots$ oraring
$a_1 + a_2 + a_3 + \cdots$
$k_1 \quad k_2$) we do the ordeig till we get $1, 2, 3, \ldots N \in \{k_1, k_2, \ldots k_{n_0}\}$

let s_n denote the partical sum for $\sum a_n$. Then for $n > n_0$ by triage inequality

$$|s_n - t_n| \leq |s_n| + |t_n| \leq \varepsilon$$

because $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t $\forall m \geq n \geq N$ $\sum_{i=n}^{m} |a_i| \leq \varepsilon$ holds and the $a_1, a_2, \ldots a$ caulle in the differene of $s_n - t_n$.

Differentiation :

Defn : let f be defined on an open interval (a, b) and assume $c \in (a, b)$ Then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

if lim exist then it is equal to $f'(c)$

Theorm 4.1 : (Diff $\Rightarrow$ cont) $f : (a, b) \to \mathbb{R}$ be a function. If f is diff at a point $c \in (a, b)$, then f is cont at c.

proof :

$$\lim_{t \to c} (f(t) - f(c)) = \lim_{t \to c} \left(\frac{f(t) - f(c)}{t - c} \right) (t - c)$$

$$= f'(c) . 0$$
$$= 0$$

Note : Converse is not true, eg : $f(x) = |x|$ is cont but not diff at 0.

Theorm 4.2 : Assume f, g are real valued funtions defined on (a, b) and diff at $c \in (a, b)$, then $f+g, f-g, f \cdot g$ are also diff at c. Also f/g is diff at c if $g(c) \neq 0$.

ⓐ $(f \pm g)'(c) = f'(c) \pm g'(c)$

ⓑ $(f \cdot g)'(c) = f'(c) g(c) + f(c) g'(c)$

③ $(f/g)'(c) = \dfrac{f'(c)g(c) - g'(c)f(c)}{(g(c))^2}$ if $g(c) \neq 0$

Proof: $\lim_{x\to c} \dfrac{(f\pm g)(x) - (f\pm g)(c)}{(x-c)}$

$= \lim_{x\to c} \dfrac{f(x)-f(c)}{x-c} \pm \dfrac{g(x)-g(c)}{x-c}$

$= f'(c) \pm g'(c)$

for product: let $h = f\cdot g$

$h(x) - h(c) = f(x)(g(x) - g(c)) + g(c)(f(x) - f(c))$

$\lim_{x\to c} \dfrac{h(x)-h(c)}{x-c} = \lim_{x\to c} f(x)\dfrac{(g(x)-g(c))}{x-c} + \lim_{x\to c} g(c)\left(\dfrac{f(x)-f(c)}{x-c}\right)$

$\lim_{x\to c} \dfrac{f\cdot g(x) - f\cdot g(c)}{x-c} = f(c)g'(c) + g(c)f'(c)$

for $h = f/g$ then

$\lim_{x\to c} \dfrac{h(x)-h(c)}{x-c} = \lim_{x\to c}\left(\dfrac{1}{x-c}\right)\left(\dfrac{f(x)}{g(x)} - \dfrac{f(c)}{g(c)}\right)$

$= \lim_{x\to c}\left(\dfrac{1}{x-c}\right)\left(\dfrac{f(x)g(c) - g(c)f(c) + g(c)f(c) - f(c)g(x)}{g(x)g(c)}\right)$

$= \lim_{x\to c}\left(\dfrac{1}{x-c}\right)\left(\dfrac{g(c)[f(x)-f(c)] - f(c)[g(x)-g(c)]}{g(x)g(c)}\right)$

$= \dfrac{1}{(g(c))^2}\left[f'(c)g(c) - f(c)g'(c)\right]$

$= \dfrac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$

Example: ① f is const $f'(c) = 0$

② for $f(x) = x$

$\dfrac{\cancel{x-c}}{\cancel{x-c}} = 1$

$\lim_{x\to c} \dfrac{f(x)-f(c)}{x-c} = 1 = f'(x)$

③ $f(x) = x^n$

$f'(x) = (x^{n-1}\cdot x)' = x^{n-1}\cdot 1 + (x^{n-1})'x$

$= x^{n-1} + x(x^{n-2}\cdot x)'$

$= x^{n-1} + x(x^{n-2} + (x^{n-2})')$

$= 2x^{n-1} + (x^{n-2})' \ldots$

$f'(x) = nx^{n-1}$

④ polynomials are differentiable by the fact that $f(x) = x^n$ is diff

Note $h(x) = f(x)/g(x)$ are also differentiable.
↘ polynomials

Note: we denote

$$f'_+(c) = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \leftarrow \text{left hand limit}$$

$$f'_-(c) = \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \leftarrow \text{right hand limit}$$

Theorm 4.3: (Chain rule)
Let $f : S \to \mathbb{R}$
↘ open intervals
$g : f(S) \to \mathbb{R}$
assume, $\exists\, c \in S$ s.t $f(c)$ is diff at c and if $g(f(c))$ is diff at $f(c)$, then $g \circ f$ is diff at c.

$$(g\circ f)'(c) = g'(f(c)) \cdot f'(c)$$

proof: let f^* be defined on (a,b) as follows:

$$f^*(x) = \frac{f(x)-f(c)}{x-c} \quad \forall\, x \neq c$$

$f^*(c) = f'(c)$ if $f'(c)$ exist, f^* is cont at c.

$$f^*(x) = \begin{cases} \dfrac{f(x)-f(c)}{x-c} & ; x \neq c \\ f'(c) & ; x = c \end{cases}$$

$$g^*(x) = \begin{cases} \dfrac{g(y)-g(b)}{y-b} & ; y \neq b = f(c) \\ g'(b) & ; y = b = f(c) \end{cases}$$

clearly g^*, f^* are cont at b, c respt.

$$f(x) - f(c) = (x-c) f^*(x) \quad \forall\, x \in S$$

$$g(y) - g(f(c)) = (y - f(c))\, g^*(y)$$

if $y = f(x)$ then

$$g(f(x)) - g(f(c)) = (f(x) - f(c))\, g^*(f(x))$$

$$\Rightarrow g(f(x)) - g(f(c)) = (x-c) f^*(x)\, g^*(f(x))$$

$$\Rightarrow \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x-c} = f^*(x)\, g^*(f(x))$$

$$= f'(c)\, g'(f(c))$$

Note: y in some open subinterval T of $f(S)$ contains $f(C)$
choose $x \in S$ s.t
$y = f(x) \in T$

Defn: A sequence $\{x_n\}_{n=1}^{\infty}$ (of $\mathbb{R}$) is said to diverge to $+\infty$, if for every $k > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t $x_n > k \;\forall n \geq n_0$

In this case we use notation, $\lim_{n\to\infty} x_n = +\infty$

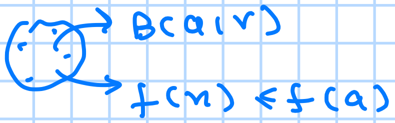
A sequence $\{x_n\}_{n=1}^{\infty}$ s.t $y_n = -x_n$, if $\{y_n\}_{n=1}^{\infty}$ (of $\mathbb{R}$) div to $+\infty$, then we say $\lim_{x\to\infty} (x_n) = -\infty$ (divg. to $-\infty$)

3rd oct:

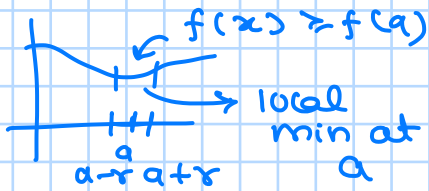
Defn: Let $f: S \to \mathbb{R}$ be a function where S is a subset of a metric space X and assume $a \in S$. Then f is said to have a <u>local maxima</u> at a if $\exists r > 0$ s.t

$$f(x) \leq f(a) \quad \forall x \in B(a,r) \cap S. \text{ If } \exists r' > 0 \text{ s.t } f(x) \geq f(a) \quad \forall x \in B(a,r') \cap S$$

then f is said to have <u>local minima</u> at a.



local maxima



<u>Theorm 4.4</u>: Assume that a function $f:(a,b) \to \mathbb{R}$ has a local min or local max at point c of (a,b). If f has a derivative at c, then $f'(c)$ must be 0.

Proof: Assume that f has local max at interior point c of (a,b)

so, $\exists r > 0$ s.t

$$a < c - r < c < c + r < b$$

and $f(x) \leq f(c) \quad \forall x \in B(c,r)$

now, $\dfrac{f(x) - f(c)}{x - c}$ for $x \in (c - r, c)$

we have

$$\frac{f(x) - f(c)}{x - c} \geq 0$$

also $\displaystyle\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$

$$\Rightarrow f'_-(c) \geq 0$$

as derivative exist $\Rightarrow f'(c) \geq 0$ ——— ①

or $f'_+(c) \geq 0$

now for $x \in (c, c+r)$

$\dfrac{f(x) - f(c)}{x - c} \leq 0$ or $\displaystyle\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$

$\Rightarrow f'_+(c) \leq 0$

$\Rightarrow f'(c) \leq 0$ ——— ②

as $f'(c) \leq 0$ and ≥ 0

$$\Rightarrow f'(c) = 0$$

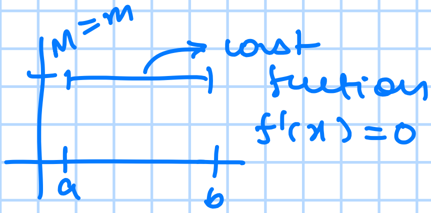
for local minima, a similar argument can be used.

<u>Theorm 4.5</u>: (Rolle's theorm)

Assume f has a derivative at each point of an interval (a,b) and assume that f is cont on $[a,b]$. If $f(a) = f(b)$, then $\exists\, c \in (a,b)$ s.t

$$f'(c) = 0$$

Proof: Let's assume f' is never zero. Since f is cont on compact set $[a,b]$ it attains it's maximum M and minimum m at some point in $[a,b]$. Neither of this are attained in (a,b) as $f'(x) \neq 0$ so the min/max are end points.



As $f(a) = f(b)$

$\Rightarrow$ f is const

or $f' = 0$ $\forall x \in$ Domain

this is a contradiction

so $\exists$ some $c \in (a,b)$ s.t

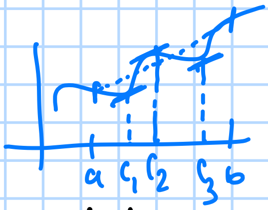
$f'(c) = 0$

Theorm 4.6 : (Mean value theorm)

If f is cont real valued function on $[a,b]$ which is diff in (a,b), then

$\exists c \in (a,b)$ s.t

$$\frac{f(a)-f(b)}{a-b} = f'(c)$$



proof : we will prove a more generalised mean value theorm

Theorm 4.7 : (generalised mean value theorm)

let f and g be cont real valued function on $[a,b]$ which are diff on (a,b)

$\exists c \in (a,b)$ s.t

$$f'(c)\big(g(b)-g(a)\big) = g'(c)\big(f(b)-f(a)\big)$$

proof :

$$h(x) = f(x)\,(g(b)-g(a)) - g(x)\,(f(b)-f(a))$$

h is cont on $[a,b]$, diff on (a,b) and

$$h(a) = f(a)\,g(b) - g(a)\,f(b)$$

$$h(b) = f(a)\,g(b) - g(a)f(b)$$

So by rolles theorm, $\exists c \in (a,b)$ s.t

$$h'(c) = 0$$

or $f'(c)[g(b)-g(a)]$

$= g'(c)[f(b) - f(a)]$

Note : putting $g(x) = x$, we get the mean value theorm from our generalised mean value theorm

Theorm 4.8 : Assume that f has a derivative at each point of an open interval (a,b) and f is cont on $[a,b]$.

(a) If $f'(x) \geq 0$, $\forall x \in (a,b)$ then f is monotonic inc on $[a,b]$

(b) If $f'(x) = 0$, $\forall x \in (a,b)$ then f is const fuction on $[a,b]$

(c) $f'(x) \leq 0$, $\forall x \in (a,b)$, then f is mon dec on $[a,b]$

proof :

let $x < y$. Applying m.v.t to sub int $[x,y]$ of $[a,b]$ we get

$$f(y) - f(x) = f'(c)\,(y-x)$$

as $f'(c) \geq 0$ (given)

$\Rightarrow$ $f(y) - f(x) \geq 0$

$\Rightarrow$ $f(y) \geq f(x)$ for $[x,y]$

$\therefore$ f is increasing

Same proof for other cases.

Note : $f^{(0)} = f$

$$f^{(1)} = \frac{d}{dx} f$$

$\vdots$

Theorm 4.9 : (Taylor) Let f be a function having finite n^{th} derivative $f^{(n)}$ everywhere in (a,b) and $f^{(n-1)}$ cont on $[a,b]$. Assume $c \in [a,b]$. Then for every $x \in [a,b]$, $x \neq c$ $\exists\, x_1$ between x and c s.t

finite taylor series

$$f(x) = f(c) + \sum_{k=1}^{n-1} \frac{f^{(k)}(c)(x-c)^k}{k!} + \underbrace{\frac{f^{(n)}(x_1)(x-c)^n}{n!}}_{\text{Reminder term}}$$

proof : we prove a more general version

Theorm 4.10 : Let f and g be two function havg finite n^{th} derivative, cont $f^{(n-1)}$, $g^{(n-1)}$ on $[a,b]$. Assume $\exists\, c \in [a,b]$ then $\forall\, x \in [a,b]$, $x \neq c$, $\exists\, x_1$ b/w x and c

s.t $$\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)(x-c)^k}{k!} \right] g^{(n)}(x_1)$$

$$= f^{(n)}(x_1) \left[g(x) - \sum_{k=0}^{n-1} \frac{g^{(k)}(c)(x-c)^k}{k!} \right]$$

proof : we discuss the case when $c < x \leq b$

keep x fixed and define functions F and G as follows:

$$F(t) = f(t) + \sum_{k=1}^{n-1} \frac{f^{(k)}(t)(x-t)^k}{k!} \quad \text{— ⓐ}$$

$$G(t) = g(t) + \sum_{k=1}^{n-1} \frac{g^{(k)}(t)(x-t)^k}{k!}$$

for each $t \in [c,x]$. Then F and G are cont on $[c,x]$ and diff on (c,x)

By theorm 4.7 :

$\exists\, x_1 \in (c,x)$ s.t

$$G'(x_1)(F(x) - F(c)) = F'(x_1)(G(x) - G(c))$$

Note : $G(x) = g(x)$
$F(x) = f(x)$

we get $G'(x_1)\left(f(x) - F(x)\right) = F'(x_1)\left(g(x) - G(c)\right)$ — ⓑ

on diff ⓐ we get, $F(t) = f(t) + \sum_{k=1}^{n-1} \frac{f^{(k)}(t)(x-t)^k}{k!}$

$$F'(t) = f'(t) + \frac{d}{dt}\left(\frac{f^{(1)}(t)}{1!}(x-t)^1 + \frac{f^{(2)}(t)}{2!}(x-t)^2 + \cdots \frac{f^{(n-1)}(t)}{(n-1)!}(x-t)^{n-1} \right)$$

$$= \cancel{f'(t)} - \cancel{\frac{f^{(1)}(t)}{1!}} + \cancel{\frac{f^{(2)}(t)}{1!}(x-t)} - \cancel{\frac{2}{2!} f^{(2)}(t)(x-t)} + \cdots \frac{f^{(n)}(t)(x-t)^{n-1}}{(n-1)!}$$

$$= \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}$$

similarly $G'(t) = \frac{g^{(n)}(t)}{(n-1)!}(x-t)^{n-1}$

now $\cancel{\frac{f^{(n)}(x_1)}{(n-1)!}(x-x_1)^{n-1}}(g(x) - G(c)) = \cancel{\frac{g^{(n)}(x_1)}{(n-1)!}(x-x_1)^{n-1}}(f(x) - F(c))$

$$\Rightarrow \left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k \right] g^{(n)}(x_1)$$

$$= f^{(n)}(x_1)\left[g(x) - \sum_{k=0}^{n-1} \frac{g^{(k)}(c)}{k!}(x-c)^k \right]$$

Note: putting $g(x) = (x-c)^n$ we get $g^{(k)}(c) = 0$ for $0 \leq k \leq n-1$ and $g^{(n)}(x) = n!$

thus prove. $f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k = \frac{f^{(n)}(x_1)(x-c)^n}{(n)!}$

or $f(x) = f(c) + \sum_{k=1}^{n-1} \frac{f^{(k)}(c)(x-c)^k}{k!} + \frac{f^{(n)}(x_1)(x-c)^n}{n!}$

special case or taylor series (finite condition)

7th oct:

Recap: IVP: $f:[a,b]\to\mathbb{R}$ cont

then as $[a,b]$ connected

$f([a,b])$ is connected

SO $f(a)<c<f(b)$

$\exists\, x\in(a,b)$

s.t $f(x)=c$

IVP in differentiation:

$f:[a,b]\to\mathbb{R}$

$f'_+(a),\ f'_-(b)$

f' has IVP (Darboux)

Theorm 4.11: (Darboux) (IVP for differentiation)

Suppose f is a real diff fuction on $[a,b]$ and suppose

$f'_+(a)<\lambda<f'_-(b)$ (other direction can be shown)

then there is a point $x\in(a,b)$

s.t $f'(x)=\lambda$

(Note: continuity of f' is not assumed)

proof: $g:[a,b]\to\mathbb{R}$

$g(x)=\dfrac{f(x)-f(a)}{x-a}$ for $x\neq a$

at a

$g(x)=f'_+(a)$ ← Right hand derivative

Note: g is continious as $x\neq a$ it is cont

and for $x\to a$ (from R side)

$\dfrac{f(x)-a}{x-a}\to f'_+(a)$

Now by IVT, g takes all values b/w

$\underset{g(b)}{\underset{\parallel}{\dfrac{f(b)-f(a)}{b-a}}}$ and $\underset{g(a)}{\underset{\parallel}{f'_+(a)}}$

now using mean value theorm

$\exists\ \xi$ s t

$g(x)=\dfrac{f(x)-f(a)}{x-a}=f'(\xi)$

$\xi\in(a,x)\subseteq(a,b)$

so f' takes every value b/w $f_+'(a)$ and $\frac{f(b)-f(a)}{b-a}$ in (a,b).

similarly, $h:[a,b]\to\mathbb{R}$

$$h(x)=\frac{f(b)-f(x)}{b-x}$$

$x\neq b$

and $h(x)=f_-'(b)$

similar to g, h is cont

so, f' takes every value b/w $\frac{f(b)-f(a)}{b-a}$ and $f_-'(b)$ in (a,b)

combining both it takes every value b/w $f_+'(a)$ and $f_-'(b)$

Theorm 4.12 : (L'Hospital Rule)

Suppose f and g are differentiable in (a,b), and $g'(x)\neq 0$, $\forall\ x\in(a,b)$ where $-\infty\leq a<b\leq+\infty$

Suppose

$$\lim_{x\to r}\frac{f'(x)}{g'(x)}=A$$

if $f(x)\to 0$ and $g(x)\to 0$ as $x\to r$

then $\lim_{x\to r}\frac{f(x)}{g(x)}=A$ for $r\in(a,b)$

proof : using the general case of mean value theorm

i.e $\exists\ c\in(a,b)$ s.t

$$f'(c)(g(b)-g(a))=g'(c)(f(b)-f(a))$$

we insted of a,b

as $\lim_{x\to r}f(x)=0$

$\Rightarrow f(r)=0$ (as f is cont)

and

$g(r)=0$ (as g is cont)

now in this case $\exists\ c\in(r,r+h)$ s.t

$$f'(c)(g(r+h)-\cancel{g(r)})$$

$$=g'(c)(f(r+h)-\cancel{f(r)})$$

$$\Rightarrow\frac{f'(c)}{g'(c)}=\frac{f(r+h)}{g(r+h)}$$

as $h\to 0$ $x\to r$

$$\lim_{x\to r}\frac{f'(x)}{g'(x)}=\lim_{x\to r}\frac{f(x)}{g(x)}$$

5. Monotonic and bounded variation functions:

Defn: A real valued f defined on S of $\mathbb{R}$

(a) increasing: if $\forall x$ and $y \in S$, $x \leq y \Rightarrow f(x) \leq f(y)$ (same for dec)

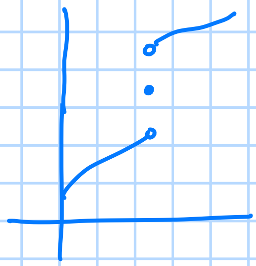
(b) strictly inc: $x < y \Rightarrow f(x) < f(y)$ (same for st. dec)

(c) f is monotonic if inc or dec function

Theorem 5.1: f is inc on $[a,b]$, for $f(c+)$ and $f(c-)$ exist for each c in (a,b) and we have

$$f(c^-) \leq f(c) \leq f(c+)$$

At endpoint $f(a) \leq f(a+)$ and $f(b^-) \leq f(b)$



proof: let $A = \{ f(x) \mid a < x < c \}$, since f is inc, the set is bounded above by $f(c)$ by comp. Let $\alpha = \sup A$

Then $\alpha \leq f(c)$

Claim: $f(c^-)$ exist and is equal to α

let $\varepsilon > 0$ be given.

we want $\exists \delta > 0$ s.t $c - \delta < x < c$

$\Rightarrow |f(x) - \alpha| < \varepsilon$

as $\alpha = \sup A$, $\exists f(x_1) \in A$ s.t

$$\alpha - \varepsilon < f(x_1) \leq \alpha$$

where $a < x_1 < c$

since f is increasing for $x \in (x_1, c)$ → see

$$\alpha - \varepsilon < f(x_1) \leq f(x) \leq \alpha$$

and so $|f(x) - \alpha| < \varepsilon$

therefore $\delta = c - x$

we get $c - \delta < x < c \Rightarrow x_1 < x < c$

$\Rightarrow |f(x) - \alpha| < \varepsilon$

so $f(c-) \leq f(c)$

similarly we can use the same argument to show

$$f(c) \leq f(c+)$$

so, $f(c^-) \leq f(c) \leq f(c^+)$

Theorem 5.2: If f is a monotone function on (a,b), then the set of discontinuities of f is countable.

Proof: Assume f is inc and E be set of all points where f is discont. associate each $x \in E$, a rational $r(x)$ s.t

$$f(x-) < r(x) < f(x+)$$

If $a < x_1 < x_2 < b$ then

$$f(x_1+) = \inf_{x_1 < t < b} f(t)$$

$$= \inf_{x_1 < t < x_2} f(t)$$

similarly $f(x_2^-) = \sup_{a < t < x_2} f(t)$

$$= \sup_{x_1 < t < x_2} f(t)$$

as $\inf_{x_1 < t < x_2} f(t) \leq \sup_{x_1 < t < x} f(t)$

$$\Rightarrow f(x_1+) \leq f(x_2-)$$

so $x_1 \neq x_2 \Rightarrow r(x_1) \neq r(x_2)$

or we have 1-1 correspondance b/w E and a subset of $\mathbb{Q}$

so E is countable ($\because E \subseteq \mathbb{Q}$)

Defn: (a) If $[a,b]$ is an interval, a set $P = \{x_0, \dots x_n\}$ s.t

$$a = x_0 < x_1 < x_2 \dots < x_n = b$$

is called a partition of $[a,b]$.

Denote

$$\Delta x_k = x_k - x_{k-1}$$

the collection of all partition of $[a,b]$ will be denoted by $P[a,b]$

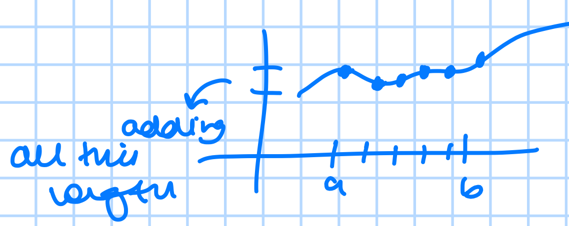
(b) let f be a function on $[a,b]$. If $P = \{x_0, \dots x_n\}$ is a partition of $[a,b]$, write

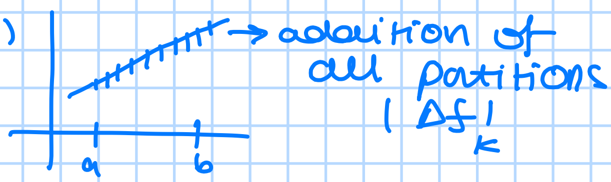
$$\Delta f_k = f(x_k) - f(x_{k-1}) \quad \forall\, k = 1,2,\dots n$$

If $\exists\, M > 0$ s.t

$$\sum_{k=1}^{n} |\Delta f_k| \leq M$$

for all partitions of $[a,b]$, then f is said to be of bounded variation.





Theorm 5.3: If f is monotonic on $[a,b]$, then f is of bounded var.

proof:

let f be inc

$\Rightarrow \Delta f_k \geq 0$

$\forall$ k and hence

$$\sum_{n=1}^{k} |\Delta f_k| = \sum_{n=1}^{k} \Delta f_k = \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))$$

$$= f(b) - f(a)$$

so f is bounded variation on $[a,b]$

same for dec.

Theorm 5.4: If f is a cont on $[a,b]$ and if f' exist and is bonded on (a,b) say $|f'(x)| \leq A$ for $x \in (a,b)$ then f is b.V on $[a,b]$.

proof: By MVT

$$\Delta f_k = f(x_k) - f(x_{k-1})$$

$$= f'(t_k)(x_k - x_{k-1})$$

for some

$t_k \in (x_{k-1}, x_k)$

$$\sum_{k=1}^{n} |\Delta f_k| \leq \sum_{k=1}^{n} |f'(t_k)| \Delta x_k \leq A \sum_{k=1}^{n} \Delta x_k = A(b-a)$$

Example: $f(x) = x^{1/2}$ is B.V but deriative is not bounded

8^{th} oct :

Recap : $\Delta x_k = x_k - x_{k-1}$

$P = \{ \underset{a}{x_0}, x_1 \dots x_{n-1}, \underset{b}{x_n} \}$

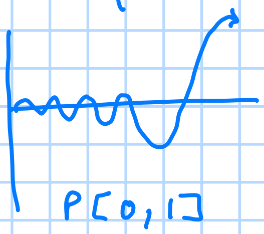
$P[a,b]$ $\quad x_{n-1} < x_n$

$\Delta f_k = f(x_k) - f(x_{k-1})$

Bounded variation : $\Sigma |\Delta f_k| \leq M$

* Monotic is B.V
* Derivative bounded $\Rightarrow$ B.V

Examples : (a) $f(x) = \begin{cases} x\cos(\pi/2x) & x \neq 0 \\ 0 & x = 0 \end{cases}$



$P[0,1]$

Not B.V

cont $\not\Rightarrow$ B.V

$P = \{0, \frac{1}{2n}, \frac{1}{2n-1}, \dots \frac{1}{3}, \frac{1}{2}, 1\}$

$\sum_{k=1}^{2n} |\Delta f_k| = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \dots |f(x_{2n}) - f(x_{2n-1})|$

$= |f(\frac{1}{2n}) - f(0)| + |f(\frac{1}{2n-1}) - f(\frac{1}{2n})| + \dots + |f(1) - f(\frac{1}{2})|$

$= \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{2n-2} + \dots \dots \frac{1}{2} + \frac{1}{2}$

$= \frac{1}{n} + \frac{1}{n-1} + \dots \dots 1$

$= \sum_{k=1}^{n} \frac{1}{k}$

not convergent so not $\leq M$.

(b) $f : [0,1] \to \mathbb{R}$, $f(x) = x^{\frac{1}{2}}$ $\quad f$ is monotone

but $f'(x) \to \infty$ as $x \to 0$

so $f'(x)$ not bounded on $(0,1)$

so $f'(x)$ not bounded $\not\Rightarrow$ not B.V

Defn: Let f be a function of bounded variation on $[a,b]$ and let $\Sigma(P)$ denote the $\sum_{k=1}^{n} |\Delta f_k|$ corresponding to $P=\{x_0,\dots,x_n\}$ of $[a,b]$.

The number $V_f(a,b) = \sup\{\Sigma(P) \mid P \in \mathcal{P}[a,b]\}$ is called total variation of f on $[a,b]$.

Theorem 5.5: If f is of bonded variation on $[a,b]$, then f is bonded on $[a,b]$

proof: for $x \in [a,b]$

$$\sum |\Delta f_k| = |f(x)-f(a)| + |f(b)-f(x)| \leq M$$

$$\Rightarrow |f(x)-f(a)| \leq M$$
$$\Rightarrow |f(x)| \leq M + |f(a)|$$

so B.V $\Rightarrow$ Bounded

Theorm 5.6: Assume f and g are b.v on $[a,b]$, then so is their sum, difference, and product. Further

$$V_{f\pm g} \leq V_f + V_g$$

$$V_{fg} \leq A V_f + B V_g$$

$$A = \sup\{|g(x)| \mid x \in [a,b]\}$$

$$B = \sup\{|f(x)| \mid x \in [a,b]\}$$

proof: (a) $q = f \pm g$

for every P of $[a,b]$

$$|\Delta q_k| = |(f(x_k) \pm g(x_k)) - (f(x_{k-1}) - g(x_{k-1}))|$$

$$|\Delta q_k| \leq |\Delta f_k| + |\Delta g_k|$$

Therefore $\sum |\Delta q_k| \leq \sum |\Delta f_k| + \sum |\Delta g_k|$

$$\leq M_f + M_g$$

so $V_{f\pm g} \leq V_f + V_g$

(b) $h(x) = f(x)\, g(x)$

$\forall P \in \mathcal{P}[a,b]$

$$|\Delta h_k| = |f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1})|$$

$$\leq |f(x_k)g(x_k) - f(x_{k-1})g(x_k)| + |f(x_{k-1})g(x_k) - f(x_{k-1})g(x_{k-1})|$$

$$\leq A|f(x_k) - f(x_{k-1})| + B|g(x_k) - g(x_{k-1})|$$

$$= A|\Delta f_k| + B|\Delta g_k|$$

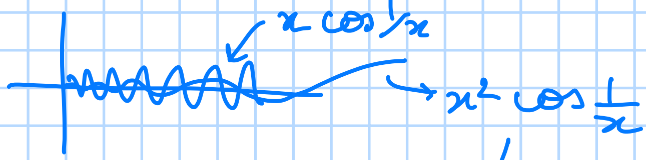
so $\sum |\Delta h_k| \leq A \sum |\Delta f_k| + B \sum |\Delta g_k|$
and
$V_{fg} \leq AV_f + BV_g$

Note: There are function of B.V st $1/f$ is not B.V

Example:

$f(x) \longrightarrow 0$
as $x \to x_0$
$1/f$ not bounded close to x_0
So $\frac{1}{f}$ not B.V (f B.V $\Rightarrow$ B) is N.B $\Rightarrow$ N.B.V)

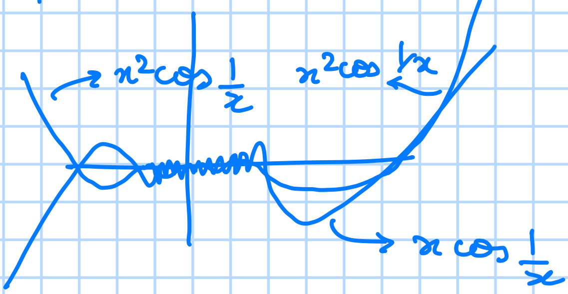
Example: $f(x) = \begin{cases} x^2\cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$



$f'(0) = 0$
$x \neq 0$
$f'(x) = \sin\left(\frac{1}{x}\right) + 2x\cos\left(\frac{1}{x}\right)$

so $|f'(x)| \leq 3$
$\forall x \in [0,1]$
so f is B.V on $[0,1]$

Note:



Theorem 5.7: f on $[a,b]$

f is b.v $\Leftrightarrow$ f can be written as diff of two inc functions on $[a,b]$

proof: V on $[a,b]$ as

$$V(x) = \begin{cases} V_f(a,x) & a < x \leq b \\ 0 & x = a \end{cases}$$

claim: V and V−f are inc on $[a,b]$

$a \leq x < y \leq b$

$V_f(a,y) = V_f(a,x) + V_f(x,y)$
$\Rightarrow V(y) = V(x) + V_f(x,y)$
$\Rightarrow V(y) - V(x) \geq 0$
$\therefore$ V is inc

also $V_f(x,y) \geq f(y) - f(x)$ with trivial partition

or $V(y) - V(x) \geq f(y) - f(x)$

$\Rightarrow V(y) - f(y) \geq V(x) - f(x)$

or $v-f$ is also inc function

$\therefore$ $v, v-f$ are inc function s.t

$$v-(v-f) = f$$

conversely if $f = g-h$ for 2 inc functions

$$\Delta f_k = \Delta g_k - \Delta h_k$$

$$\Rightarrow |\Delta f_k| = |\Delta g_k - \Delta h_k| \leq |\Delta g_k| + |\Delta h_k|$$

so $\sum |\Delta f_k| \leq M_g + M_h$

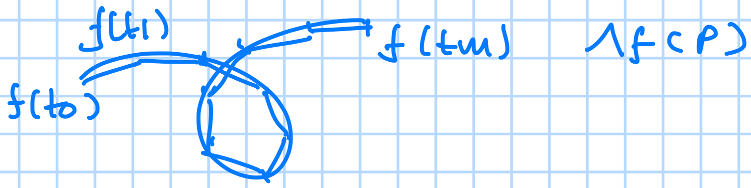
Def^n: (a) A cont function $f:[a,b] \to \mathbb{R}^n$ is called a path.

(b) Let $f:[a,b] \to \mathbb{R}^n$ be a path in $\mathbb{R}^n$. For any $P \in \mathcal{P}[a,b]$

$$P = \{t_0, \dots, t_m\}$$

$$\Lambda_f(P) = \sum_{k=1}^{m} \| f(t_k) - f(t_{k-1}) \|$$

$$\|(x_1, x_2, \dots x_n)\| = \left(\sum |x_i|^2\right)^{1/2}$$



(c) set $\{\Lambda_f(P) \mid P \in \mathcal{P}[a,b]\}$ is bounded, then the path is said to be rectifiable

(left from here)

(see lect 8 and lect 9 from prof)

14th oct:

6. Integration:

let $f:[a,b]\to\mathbb{R}$ be bounded. let $P=\{x_0,x_1,\dots x_n\}$ of $[a,b]$ be a partition.

$$m_i(f):=\inf\{f(x)\mid x\in[x_{i-1},x_i]\}$$

$$M_i(f):=\sup\{f(x)\mid x\in[x_{i-1},x_i]\}$$

$\int_{-\infty}^{\infty} f(x)\,dx$ Not this

$\int_0^1 \frac{1}{x}\,dx$ Not this

Rimenn

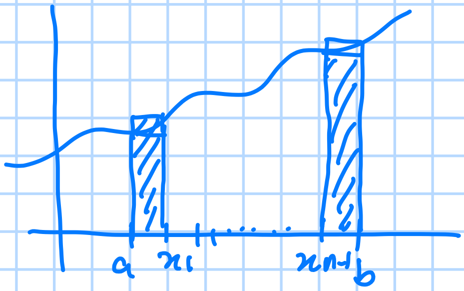
$f:[a,b]\to\mathbb{R}$ bounded

to $\int_a^b f(x)\,dx$

Now: $m(f)=\inf\{f(x)\mid x\in[a,b]\}$

$M(f)=\sup\{f(x)\mid x\in[a,b]\}$

Lower sum and upper sum:

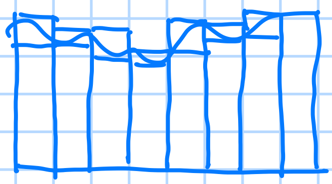


upper sum $=\sum M_i(x_i-x_{i-1})=U(P,f)$

lower sum $=\sum m_i(x_i-x_{i-1})=L(P,f)$

$$L(P,f):=\sum m_i(f)(x_i-x_{i-1})$$

$$U(P,f):=\sum M_i(f)(x_i-x_{i-1})$$



Idea is to refine partitions

prop 6.1: let $f:[a,b]\to\mathbb{R}$ be a bounded function, then for any partition P of $[a,b]$ we have

$$m(f)(b-a)\leq L(P,f)\leq U(P,f)\leq M(f)(b-a)$$

proof: $P=\{x_0,\dots x_n\}$

$$m(f)\leq m_i(f)\leq M_i(f)\leq M(f)\quad\forall i=1,2,\dots n$$

$$L(P,f)=\sum m_i(f)(x_i-x_{i-1})$$

$$\geq m(f)\sum(x_i-x_{i-1})$$

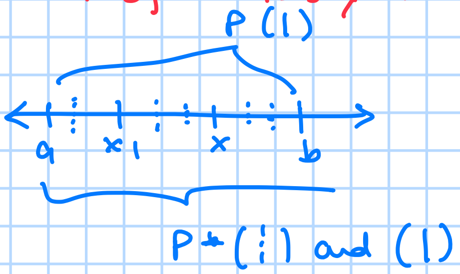
$$\Rightarrow m(f)(b-a)\leq \underbrace{L(P,f)\leq U(P,f)}_{\text{same thing here}}\leq M(f)(b-a)$$

Define $L(f) = \sup\{L(P,f) \mid P \text{ is a partion of } [a,b]\}$

$U(f) = \inf\{U(P,f) \mid P \text{ is a partion of } [a,b]\}$

Lower and upper (Riemann) integrals respectively

Refinement: given P of $[a,b]$, P^* is a refinement if $\forall x \in P$ is also in P^*.



Lemma 6.2 : $f:[a,b] \to \mathbb{R}$ Bounded

(i) P is partion of $[a,b]$ ← Refinement

$L(P,f) \leq L(P^*,f)$

$U(P,f) \geq U(P^*,f)$

Consequenty $U(P^*,f) - L(P^*,f) \leq U(P,f) - L(P,f)$

(ii) $P_1, P_2 \in P[a,b]$ then

$$L(P_1,f) \leq U(P_2,f)$$

(iii) $L(f) \leq U(f)$

proof : (i) $P = \{x_1, x_2 \ldots x_n\} \in P[a,b]$

P^* be a refines with 1 extra point

$x^* \in (a,b)$ s.t $\quad x_{i-1} < x^* < x_i$
for some $i \in \{1,2,\ldots n\}$

$M_l^* = \sup\{f(x) \mid x \in [x_{i-1}, x^*]\}$

$M_r^* = \sup\{f(x) \mid x \in [x^*, x_i]\}$

$M_l^* \leq M_i(f)$

$M_r^* \leq M_i(f)$

so

$U(P,f) - U(P^*,f)$

$= M_i(f)(x_i - x_{i-1})$

$- M_L^*(x^* - x_{i-1}) - M_r^*(x_i - x^*)$

$\geq M_i(f)(x_i - x_{i-1})$

$-M_i(f)(x^* - x_{i-1})$
$-M_i(f)(x_i - x^*)$
≥ 0

$\Rightarrow \quad U(P, f) \geq U(P^*, f)$

If P^* has more than 1, we repeate the step to get same

$$U(P^*, f) \leq U(P, f) \quad \text{—— (a)}$$
$$L(P, f) \leq L(P^*, f) \quad \text{—— (b)}$$
$$\Rightarrow U(P^*, f) - L(P^*, f)$$
$$\leq U(P, f) - L(P, f)$$

(ii) P^* denote the common refinement i.e union of partitions P_1, P_2

then from (i)

$$L(P_1, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P_2, f)$$
$$\Rightarrow L(P_1, f) \leq U(P_2, f)$$

(iii) Let us fix a partition P' of $[a, b]$

By (ii) we have

$$L(P', f) \leq U(P, f) \quad \forall P \in P[a, b]$$

hence $L(P', f)$ is a lower bound of the set

$$\{ U(P, f) \mid P \text{ is a partion of } [a, b] \}$$

$$\Rightarrow L(P', f) \leq U(f) = \inf \{ U(P, f) \mid P \text{ is a partion of } [a, b] \}$$

as P' is arbitary partition

$$\Rightarrow \sup \{ L(P', f) \mid P' \in P[a, b] \} \leq U(f)$$
$$\Rightarrow L(f) \leq U(f)$$

now, $\frac{L(f)}{U(f)} \rightarrow$ exist and also ≤ 1

Defn: If $f : [a, b] \rightarrow \mathbb{R}$ be bounded function, then f is said to be <u>integrable</u> (on $[a, b]$) if

$$L(f) = U(f)$$

in this case the common value $L(f) = U(f)$ is called (Riemann) <u>integral</u> of f and is denoted by

$$\int_a^b f(x)\,dx$$

Propn 6.3: (Basic inequality for Riemann integrals) Suppose $f:[a,b]\to\mathbb{R}$ is an integrable function & $\exists\, \alpha,\beta\in\mathbb{R}$

s.t $\beta \leq f(x) \leq \alpha$, $\forall\, x\in[a,b]$ then

$$\beta(b-a) \leq \int_a^b f(x)\,dx \leq \alpha(b-a)$$

proof: as $\beta \leq f(x) \leq \alpha \;\forall\, x\in[a,b]$

$$\Rightarrow \beta \leq m(f) \;\&\; M(f) \leq \alpha$$

$$\Rightarrow \beta(b-a) \leq m(f)(b-a) \leq L(P,f) \leq U(P,f) \leq M(f)(b-a) \leq \alpha(b-a)$$

$$\Rightarrow \beta(b-a) \leq m(f)(b-a) \leq L(f) = U(f) \leq M(f)(b-a) \leq \alpha(b-a)$$

$$\Rightarrow \beta(b-a) \leq \int_a^b f(x)\,dx \leq \alpha(b-a)$$

Remark: It follows that $|f| \leq \alpha$

$$\Rightarrow \left|\int_a^b f(x)\,dx\right| \leq \alpha(b-a)$$

Examples:

(i) Dirichlet function on $[a,b]$

$$f(x) = \begin{cases} 1 & x\in[a,b]\cap\mathbb{Q} \\ 0 & x\in[a,b]\cap(\mathbb{R}\setminus\mathbb{Q}) \end{cases}$$

Since each $[x_{i-1},x_i]$ contains rational number and an irrational number, we get

$$m_i(f) = 0$$
$$M_i(f) = 1 \quad \forall\, i=1,2,\dots n$$

$$L(P,f) = 0$$
$$U(P,f) = \Sigma 1(x_i - x_{i-1}) = b-a$$

$$\Rightarrow L(f) = 0$$
$$\Rightarrow U(f) = b-a$$

so as $L(f) \neq U(f)$

function is not Riemann integrable

(ii) $f(x)=1$ is integrable, any partition

$m_i = 1$, $M_i = 1$, $\Sigma m_i(x_i - x_{i-1}) = b-a = \Sigma M_i(x_i - x_{i-1})$

15^{th} oct :

prop 6.4 : (Riemann condition)
let $f:[a,b]\to\mathbb{R}$ be a bounded function.
Then f is integrable iff $\forall \varepsilon>0$, $\exists$ a partition P_ε of $[a,b]$ s.t

$$U(P_\varepsilon,f)-L(P_\varepsilon,f)<\varepsilon \quad \text{——} ⓐ$$

proof : ($\Leftarrow$) suppose ⓐ holds. Let $\varepsilon>0$, then $\exists P_\varepsilon$ partition s.t

$$\varepsilon > U(P_\varepsilon,f)-L(P_\varepsilon,f)\geq U(f)-L(f)\geq 0$$

$$\Rightarrow U(f)=L(f)$$
$$\Rightarrow f \text{ is integrable}$$

($\Rightarrow$) conversely, let's assume f is integrable then, let $\varepsilon>0$ be given.

By defn of sup and inf, $\exists$ partition Q_ε, Q_ε' s.t

$$U(Q_\varepsilon,f)<U(f)+\varepsilon/2 \quad \left(\forall a>s, a\in\mathbb{R}, \exists x\in S \Rightarrow x<a\right)$$

$$L(Q'_\varepsilon,f)>L(f)-\varepsilon/2 \quad \left(\forall a<s, a\in\mathbb{R}\Rightarrow \exists x\in S' \text{ s.t } a<x\leq s\right)$$

let P_ε = common refinement of Q_ε and Q_ε'
By part (1) of lemma 6.2 we have

$$L(f)-\varepsilon/2<L(Q'_\varepsilon,f)\leq L(P_\varepsilon,f)$$

and also

$$U(f)+\varepsilon/2>U(Q_\varepsilon,f)\geq U(P_\varepsilon,f)$$

$$\Rightarrow \quad L(f)<L(P_\varepsilon,f)+\varepsilon/2$$
$$-U(f)<-U(P_\varepsilon,f)+\varepsilon/2$$

$$\Rightarrow L(f)-U(f)<L(P_\varepsilon,f)-U(P_\varepsilon,f)+\varepsilon$$

$$\Rightarrow U(P_\varepsilon,f)-L(P_\varepsilon,f)<U(f)-L(f)+\varepsilon$$

as $U(f)=L(f)$

$$\Rightarrow U(P_\varepsilon,f)-L(P_\varepsilon,f)<\varepsilon$$

Prop 6.5 : let $f:[a,b]\to\mathbb{R}$ be a bounded function and let $c\in(a,b)$. Then f is integrable on $[a,b]$ iff f is integrable on $[a,c]$ and $[c,b]$
also $$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$$

proof :
($\Rightarrow$) If f is integrable on $[a,b]$. Let $\varepsilon>0$ then $\exists$ a partition P_ε of $P[a,b]$ s.t

$$U(P_\varepsilon,f)-L(P_\varepsilon,f)<\varepsilon$$

by theorem 6.4, let P_ε^* be a refinement of P_ε by taking additional c in the point

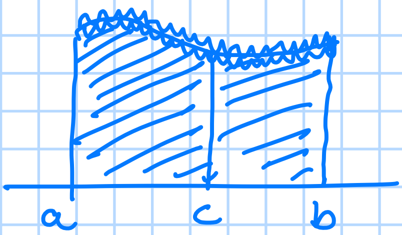
so assume $P_\varepsilon^*=\{x_1,\dots,x_n\}$
$x_k=c$
some

By part (i) of Lemma 6.2 we have

$$U(P_\varepsilon^*, f) - L(P_\varepsilon^*, f) < \varepsilon$$

now $Q_\varepsilon^* = \{x_0, x_1 \ldots x_k\} \to$ partition of $[a,c]$

$g = f|_{[a,c]}$ i.e. f is restricted on $[a,c]$.



Then $U(Q_\varepsilon^*, g) - L(Q_\varepsilon^*, g)$

$$= \sum_1^k (M_i(g) - m_i(g))(x_i - x_{i-1})$$

$$= \sum_1^k (M_i(f) - m_i(f))(x_i - x_{i-1})$$

$$\leq \underbrace{\sum_1^n (M_i f - m_i f)(x_i - x_{i-1})}_{<\varepsilon}$$

$\Rightarrow U(Q_\varepsilon^*, g) - L(Q_\varepsilon^*, g) < \varepsilon$

$\Rightarrow g = f|_{[a,c]}$ is integrable

$\Rightarrow f$ is riemann integrable for $[a,c]$

Similarly one can show for $[c,b]$

($\Leftarrow$) conversly, f is integrable on $[a,c]$ and $[c,b]$

Let $g = f|_{[a,c]}$ $h = f|_{[c,b]}$

denote the restrictions of f to $[a,c]$, $[c,b]$.

Let $\varepsilon > 0$, then by theorem 6.4,

$\exists\, Q_\varepsilon$ on $[a,c]$ s.t

$U(Q_\varepsilon, g) - L(Q_\varepsilon, g) < \varepsilon/2$

and $\exists\, R_\varepsilon$ on $[c,b]$ s.t

$U(R_\varepsilon, h) - L(R_\varepsilon, h) < \varepsilon/2$

$$\Rightarrow \left[U(Q_\varepsilon, g) + U(R_\varepsilon, h)\right] - \left[L(Q_\varepsilon, g) + L(R_\varepsilon, h)\right] < \varepsilon$$

for P_ε = union of Q_ε and R_ε points

$U(P_\varepsilon, f) = U(Q_\varepsilon, g) + U(R_\varepsilon, h)$

& $L(P_\varepsilon, f) = L(Q_\varepsilon, g) + L(R_\varepsilon, h)$

$\Rightarrow U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$

$\Rightarrow f$ is riemann integrable on $[a,b]$

now, $\underbrace{U(f, P_\varepsilon)}_{=k} = U(f, Q_\varepsilon) + U(f, R_\varepsilon)$

$\ell = L(P_\varepsilon, f)$

Clearly $l \leq \int_a^b f(x)dx \leq k$

and also

$$l = L(Q_\varepsilon, f) + L(R_\varepsilon, f)$$

or $$l \leq \int_a^c f(x)dx + \int_c^b f(x)dx \leq k$$

from these inequalities we get

$$\left|\int_a^c f(x)dx + \int_c^b f(x)dx - \int_a^b f(x)dx\right| \leq k - l < \varepsilon$$

$$\Rightarrow \left|\left(\int_a^c f(x)dx + \int_c^b f(x)dx\right) - \left(\int_a^b f(x)dx\right)\right| < \varepsilon$$

$$\Rightarrow \int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

Because, $\varepsilon > 0$ is arbitrary

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

Note :

$a = b$ then $\int_a^b f(x)dx = 0$

$a > b$ then $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Propn 6.6 : (i) If $f : [a,b] \to \mathbb{R}$ is a monotonic function, then f is integrable.

(ii) If $f : [a,b] \to \mathbb{R}$ is const. then f is integrable

proof: (i)

wlog monotonically inc

$\Rightarrow f$ is Bounded

$\Rightarrow \forall P \in P[a,b]$

say $P = \{x_0, \ldots x_n\}$

$\Rightarrow M_i(f) = f(x_i)$

$m_i(f) = f(x_{i-1}) \quad \forall\ i = 1 \ldots n$

so $$U(P,f) - L(P,f) = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})](x_i - x_{i-1})$$

if $f(a) = f(b)$

then $U(P,f) = L(P,f)$

Otherwise take $(x_i - x_{i-1}) < \dfrac{\varepsilon}{f(b) - f(a)}$

then $U(P,f) - L(P,f) < \sum [f(x_i) - f(x_{i-1})] \dfrac{\varepsilon}{f(b) - f(a)}$

$\Rightarrow U(P,f) - L(P,f) < \varepsilon$

17th oct :

recap: monotonic function is integrable as for P_ε s.t $(x_i - x_{i-1}) < \frac{\varepsilon}{f(b)-f(a)}$
(Riemann condition used)

propn 6.6 : (ii) if $f:[a,b] \to \mathbb{R}$ is cont. then f is integrable

proof : Here as $f:[a,b] \to \mathbb{R}$ is cont. then

$$\forall \varepsilon>0, \ \forall p \in [a,b], \ \exists \delta>0 \text{ s.t } |x-p|<\delta \Rightarrow |f(x)-f(p)|<\varepsilon$$

now as f is bounded.

f is uniform cont
then

$\exists \delta>0$ s.t $\forall x, y \in [a,b]$
s.t $|x-y|<\delta \Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{b-a}$

now using Riemann condition
if $\exists P_\varepsilon \in \mathcal{P}[a,b]$
s.t
$U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon \Rightarrow f$ is integrable

now let $x_i - x_{i-1} < \delta$
then
$P_\varepsilon = \{x_0, x_1, \ldots x_n\}$
so $\forall x, y \in [x_{i-1}, x_i]$
we have
$f(x)-f(y) < \frac{\varepsilon}{b-a}$

$$\Rightarrow M_i(f) - m_i(f) < \frac{\varepsilon}{b-a}$$

$$\Rightarrow \sum [M_i(f) - m_i(f)][x_i - x_{i-1}] < \left(\frac{\varepsilon}{\cancel{b-a}}\right)\left(\cancel{b-a}\right) = \varepsilon$$

$$\Rightarrow U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$$

prop 6.7 : $f, g : [a,b] \to \mathbb{R}$
integrable

(i) $f+g$ is also int
$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(ii) rf is also int $\forall r \in \mathbb{R}$
$$\int_a^b rf\,dx = r\int_a^b f\,dx$$

(iii) fg is int

proof :
(i) let $\varepsilon>0$, by riemann condition
$\exists$ Q & $R \in P[a,b]$
s.t $U(Q,f) - L(Q,f) < \varepsilon$
& $U(R,g) - L(R,g) < \varepsilon$

let P = common refinement of Q and R
by lemma 6.2 (i), we have

$$U(P,f) - L(P,f) \leq \underbrace{U(Q,f) - L(Q,f)}_{<\varepsilon}$$

$$U(P,g) - L(P,g) \leq U(R,g) - L(R,g) < \varepsilon$$

Let $P = \{x_0, x_1, \ldots, x_n\}$
also $M_i(f+g) \leq M_i(f) + M_i(g)$
$m_i(f+g) \geq m_i(f) + m_i(g)$
$\forall i = 1, 2 \ldots n$

then $\sum M_i(f+g)(\Delta x_i) \leq \sum (M_i(f) + M_i(g))(\Delta x_i)$

$U(P, f+g) \leq U(P,f) + U(P,g)$

and similarly $L(P, f+g) \geq L(P,f) + L(P,g)$

$$\Rightarrow \underbrace{U(P, f+g)}_{A} - \underbrace{L(P, f+g)}_{B} \leq (U(P,f) - L(P,f)) + (U(P,g) - L(P,g)) < 2\varepsilon$$

so $f+g$ is integrable when f, g are riemann integrable

$A = U(P,f) + U(P,g)$
$B = L(P,f) + L(P,g)$

then $B \leq L(P, f+g) \leq \int_a^b (f+g)(x)\,dx \leq U(P, f+g) \leq A$

$f \quad B \leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \leq A$

$$\Rightarrow \left| \int_a^b (f+g)(x)\,dx - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| \leq A - B < 2\varepsilon$$

$$\therefore \int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(ii) f int $\Rightarrow \gamma f$ is int

if $\gamma > 0$
$\Rightarrow M_i(\gamma f) = \gamma M_i(f)$
$\Rightarrow L(P, \gamma f) = \gamma L(P, f)$

if $\gamma < 0$
$\Rightarrow M_i(\gamma f) = \gamma m_i(f)$
$\Rightarrow L(P, \gamma f) = \gamma U(P, f)$

if $\gamma = 0$ (trivial)

for $\underline{\gamma > 0}$:
$L(P, \gamma f) = \gamma L(P,f)$
$U(P, \gamma f) = \gamma U(P,f)$
$\therefore L(\gamma f) = \gamma L(f)$
$\gamma U(f) = \gamma L(f)$
$\gamma U(f) = U(\gamma f)$

same for $\gamma < 0$

$\therefore \gamma f$ is also riemann integrable and
also $\gamma L(f) = U(\gamma f) = L(\gamma f) = \gamma U(f)$

$$\Rightarrow \int_a^b \gamma f(x)\,dx = \gamma \int_a^b f(x)\,dx$$

(iii) <u>optional</u>

f, g is integrable
fg is also integrable

<u>Note</u>: For $f:[a,b] \to \mathbb{R}$, let $|f|:[a,b] \to \mathbb{R}$ denote $|f|(x) = |f(x)|$

$$f(x) = \begin{cases} 1 ; & x \in \mathbb{Q} \\ -1 ; & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

$|f|(x) = 1 \to$ integrable
but f is <u>not</u> integrable

<u>propn 6.8</u>: $f, g : [a,b] \to \mathbb{R}$, integrable

(i) $f \leq g$ then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

(ii) $|f|$ is integrable &

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f|(x)\,dx$$

<u>proof</u>: (i) as $f \leq g$ over $[a,b]$ & let $P \in \mathcal{P}[a,b]$

$U(P,f) \leq U(P,g)$ for any partition as $M_i(f) \leq M_i(g)$

$$\Rightarrow \int_a^b f(x)\,dx = U(f) \leq U(g) = \int_a^b g(x)\,dx$$

(ii) let $\varepsilon > 0$. By Riemann condition, $\exists\ P_\varepsilon \in \mathcal{P}[a,b]$ s.t

$$U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$$

let $P_\varepsilon = \{x_0, \ldots x_n\}$
for $i = 1, 2, \ldots n$
and
$x, y \in [x_{i-1}, x_i]$

we have

$$|f|(x) - |f|(y) \leq |f(x) - f(y)| \leq M_i(f) - m_i(f)$$

$$\Rightarrow |f|(x) - |f|(y) \leq M_i(f) - m_i(f)$$

$$\Rightarrow \sum \underbrace{M_i(|f|) - m_i(|f|)}_{(\Delta x_i)} \leq \sum \underbrace{M_i(f) - m_i(f)}_{(\Delta x_i)}$$

$$\Rightarrow U(P_\varepsilon, |f|) - L(P_\varepsilon, |f|) \leq U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$$

$\therefore$ By riemann condition $|f|$ is integrable

further $-|f|(x) \leq f(x) \leq |f|(x)$

$$\Rightarrow -\int_a^b |f|(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f|(x)\,dx$$

using prop 6.7(ii)

$$\Rightarrow \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f|(x)\,dx$$

Theorm 6.9: let $f:[a,b] \to \mathbb{R}$ be integrable, $F:[a,b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,dt$$

then ① F is cont on $[a,b]$

② F satisfies lipschitz condition on $[a,b]$

③ $\exists \alpha > 0$ s.t

$$|F(x) - F(y)| < \alpha |x-y| \quad \forall x, y \in [a,b]$$

proof: f is integrable

$\Rightarrow f$ is bounded on $[a,b]$

i.e $\exists \alpha > 0$ s.t

$$|f(t)| \leq \alpha \quad \forall \quad t \in [a,b]$$

let $c \in [a,b]$ then

$\forall x \in [a,b]$

$$F(x) - F(c) = \int_a^x f(t)\,dt - \int_a^c f(t)\,dt$$

$$= \int_c^x f(t)\,dt \quad \left(\text{should be shown in two cases: } x \leq c \;,\; x > c\right)$$

$$\therefore \; |F(x) - F(c)| \leq \left| \int_c^x f(t)\,dt \right| \leq \alpha |x-c|$$

$\therefore$ F is cont, satisfies lipschitz condition.

21st oct :

Riemann integration $\int_a^b f(x)\,dx$; $f(x) = x^2$

Theorm 6.9 :

let $f:[a,b] \to \mathbb{R}$ be integrable and let $F:[a,b] \to \mathbb{R}$ be defined as $F(x) = \int_a^x f(t)\,dt$

then ① F is cont on $[a,b]$

② F satisfies Lipschitz cond on $[a,b]$

$\therefore \exists\ \alpha > 0$ s.t $|F(x) - F(y)| \leq \alpha |x-y|$ $\forall\ x, y \in [a,b]$

Example : 0 —— 1

$$\int_0^1 f(x)\,dx = U(P,f) = \sum M_i(f)(x_i - x_{i-1})$$

$$= \sum \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum i^2$$

$$= \frac{1}{n^3} \frac{(n)(n+1)(2n+1)}{6}$$

as $n \to \infty$ $U(f, P_n) \to \frac{1}{3}$

similary $L(f, P_n) \to 3$, $\therefore L(f) = U(f)$

$$= \int_a^b f(x)\,dx = \frac{1}{3}$$

proof : f is int $\Rightarrow$ f is bounded on $[a,b]$

$\exists\ \alpha > 0$ s.t

$|f| < \alpha$

and then we solve

Note : Integration is a smoothing process

Defn : A funtion $f:[a,b] \to \mathbb{R}$ is said to be anti derivative on $[a,b]$ if $\exists$ a diff funtion $F:[a,b] \to \mathbb{R}$ s.t

$$f = F'$$

here, F is called anti-deri of f.

propn 6.10 : (Fundamental theorm of calculus)

let $f:[a,b] \to \mathbb{R}$ be integrable

(i) If f has an antiderivative F, then

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

(motivation : $\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$)

(ii) let $F:[a,b] \to \mathbb{R}$ be $F(x) = \int_a^x f(t)\,dt$. If f is cont at $c \in [a,b]$, then F is an antiderivative of f on $[a,b]$

proof : f is integrable

(i) $\exists\ F$ s.t $F' = f$ then

claim : $\int_a^x f(t)\,dt = F(x) - F(a)$

case I) $x = a$:

$$\int_a^a f(t)\,dt = 0 = F(a) - F(a) \quad \text{(trivial)}$$

Case II > $x > a$: $x \in (a, b]$

then let $g = f|_{[a,x]}$ → Restriction of f on $[a,x]$

by prop 6.5 g is also integrable

let $\varepsilon > 0$, be given

By Riemann condition $\exists$ a partition $P_\varepsilon = \{x_0, \dots, x_n\}$ of $[a,x]$

s.t $U(P_\varepsilon, g) - L(P_\varepsilon, g) < \varepsilon$

By M.V.T $\exists$ $y_i \in (x_{i-1}, x_i)$ $\forall$ $i = 1, \dots n$

(we want to show $\left| F(x) - F(a) - \int_a^x f(t)\,dt \right| < \varepsilon$)

$F(x_i) - F(x_{i-1}) = F'(y_i)(x_i - x_{i-1})$ [By M.V.T]

$\Rightarrow F(x_i) - F(x_{i-1}) = f(y_i)(x_i - x_{i-1})$

$\Rightarrow F(x_i) - F(x_{i-1}) = g(y_i)(x_i - x_{i-1})$

now $L(P_\varepsilon, g) = \sum m_i(g)(x_i - x_{i-1})$

we have

$m_i(g) \leq g(y_i) \leq M_i(g)$

$\Rightarrow L(P_\varepsilon, g) \leq F(x) - F(a) \leq U(P_\varepsilon, g)$

& $L(P_\varepsilon, g) \leq \int_a^x f(t)\,dt \leq U(P_\varepsilon, g)$

$$\Rightarrow \left| F(x) - F(a) - \int_a^x f(t)\,dt \right| \leq \underbrace{\left| U(P_\varepsilon, g) - L(P_\varepsilon, g) \right|}_{< \varepsilon}$$

$$\Rightarrow \int_a^x f(t)\,dt = F(x) - F(a)$$

(ii) $F(x) = \int_a^x f(t)\,dt$ where f is cont at $c \in [a,b]$.

let $\varepsilon > 0$ be given.

so, $\exists \delta > 0$ s.t $t \in [a,b]$ and $|t - c| < \delta$

$\Rightarrow |f(t) - f(c)| < \varepsilon$

let $x \in [a,b]$, $x \neq c$ and $|x - c| < \delta$

$$\text{then } \frac{F(x) - F(c)}{x - c} = \frac{1}{x - c} \int_c^x f(t)\,dt$$

$$= \frac{1}{x - c} \left[\int_c^x f(t) - f(c)\,dt + \int_c^x f(c)\,dt \right]$$

$$\Rightarrow \left| \frac{F(x) - F(c)}{x - c} - f(c) \right| \leq \frac{1}{|x - c|} \varepsilon |x - c|$$

$\therefore F'(c) = f(c)$

or F is antiderivative of f.

Prop 6.11 : (F.T of R. I)
let $F:[a,b]\to\mathbb{R}$ be a function. Then F is diff & F' is cont iff $\exists$ a cont function $f:[a,b]\to\mathbb{R}$ s.t

$$F(x) = F(a) + \int_a^x f(t)\,dt$$

in this case $F'(x) = f(x) \;\forall x \in [a,b]$

proof : suppose F is diff and F' is cont on $[a,b]$
($\Rightarrow$) then F' is integrable
& F is the antiderivative

by FTC
$$\int_a^x F'(t)\,dt = F(x) - F(a) \quad \forall x \in [a,b]$$

by putting $f = F'$ we get our assertion

$\therefore \exists$ a cont function $f = F'$, we get
$$F(x) = F(a) + \int_a^x f(t)\,dt$$

for second part

($\Leftarrow$) $F(x) = F(a) + \int_a^x f(t)\,dt$ is given
f is cont

claim : F is diff & F' is cont
from previous theorem

$$G(x) = F(x) - F(a) = \int_a^x f(t)\,dt$$

$G'(x) = f(x)$ (from previous theorem second part)

$\Rightarrow F'(x)$ is cont
$\Rightarrow F$ is diff

Prop 6.12 : (Integration by parts)
let f be a diff function s.t f' is integrable. suppose g is integrable and has antiderivative G
then

$$\int_a^b f g\,dx = f(b)G(b) - f(a)G(a) - \int_a^b f'(x)G(x)\,dx$$

proof : let $H = fG$
$H' = fg + f'G$

now since f & G are diff, they are cont & hence integrable. Also since f' & g are given

$$\int_a^b H'(t)\,dt = H(b) - H(a)$$
$$= f(b)G(b) - f(a)G(b)$$
$$= \int_a^b fg\,dt + \int_a^b f'G\,dt$$

$$\int_a^b fg\,dt = f(b)G(b) - f(a)G(a) - \int_a^b f'G\,dt$$

prop 6.13 : (Integration by substitution)

let $\phi : [\alpha, \beta] \to \mathbb{R}$ be a diff function s.t ϕ' is integrable on $[\alpha, \beta]$ and let $\phi([\alpha,\beta]) = [a,b]$

(i) if $f : [a,b] \to \mathbb{R}$ is cont then $(f \circ \phi)\phi' : [\alpha, \beta] \to \mathbb{R}$ is int &

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f(\phi(t))\,\phi'(t)\,dt$$

(ii) $f : [a,b] \to \mathbb{R}$ is integrable & $\phi'(t) \neq 0 \quad \forall t \in (\alpha, \beta)$ then

$(f \circ \phi)|\phi'| : [\alpha, \beta] \to \mathbb{R}$ is integrable

(using darboux)

$$\& \int_{a}^{b} f(x)\,dx = \int_{\alpha}^{\beta} f(\phi(t))\,|\phi'(t)|\,dt$$

22^{nd} oct :

prop 6.13 : (Integration by substitution)
let $\phi : [\alpha, \beta] \to \mathbb{R}$ be a diff function s.t ϕ' is integrable on $[\alpha, \beta]$ and let $\phi([\alpha,\beta]) = [a,b]$

(i) if $f : [a,b] \to \mathbb{R}$ is cont
then $(f \circ \phi)\phi' : [\alpha, \beta] \to \mathbb{R}$
is int &

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f(\phi(t))\,\phi'(t)\,dt$$

(ii) $f : [a,b] \to \mathbb{R}$ is integrable & $\phi'(t) \neq 0 \quad \forall t \in (\alpha, \beta)$
then
$(f \circ \phi)|\phi'| : [\alpha,\beta] \to \mathbb{R}$ is integrable

$$\& \int_a^b f(x)dx = \int_{\alpha}^{\beta} f(\phi(t))\,|\phi'(t)|\,dt$$

proof : (i) f is cont then
for $x \in [a,b]$

$$F(x) = \int_a^x f(u)du \quad \text{from f.t.c } F \text{ is diff}$$

note $F' = f$

now set $H : [\alpha,\beta] \longrightarrow \mathbb{R}$
by $H = F \circ \phi$
then

$$H'(t) = F'(\phi(t))\,\phi'(t)$$
$$= f(\phi(t))\,\phi'(t) \qquad \forall t \in [\alpha, \beta]$$

here $f \circ \phi$ is cont and hence integrable.
Also ϕ' is integrable

$\Rightarrow$ Product H' is integrable

$$\int_{\alpha}^{\beta} f(\phi(t))\,\phi'(t)\,dt = H(\beta) - H(\alpha)$$
$$= (F \circ \phi)(\beta) - (F \circ \phi)(\alpha)$$
$$= \int_a^{\phi(\beta)} f(x)dx - \int_a^{\phi(\alpha)} f(x)dx$$
$$= \int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx$$

$$\left(F(\phi(x)) = \int_a^{\phi(x)} f(t)dt \right)$$

(ii) let $f : [a,b] \longrightarrow \mathbb{R}$ to be integrable and $\phi'(t) \neq 0 \ \forall t \in (\alpha, \beta)$
define
$\psi := (f \circ \phi)|\phi'|$
it is not known that $(f \circ \phi)|\phi'|$ is integrable

claim : $L(f) \leq L(\psi) \leq U(\psi) \leq U(f)$
and so it follows ψ is integrable
By IVP ϕ' is either >0 or
(Darboux) <0
$\forall t \in (\alpha, \beta)$

wlog :
$\phi'(t) > 0 \quad \forall\, t \in (\alpha, \beta)$
then ϕ is strictly inc on $[\alpha, \beta]$, therefore $\phi(\alpha) = a$ and $\phi(\beta) = b$.

$P = \{x_0, \dots x_n\}$
of $[a, b]$ and let $t_i = \phi^{-1}(x_i)$ for $i = 1 \dots n$
then
$\alpha = t_0 < t_1 \dots\dots < t_n = \beta$ & by part (i) of FTC

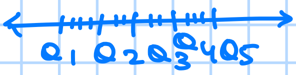
$$\int_{t_{i-1}}^{t_i} |\phi'(t)|\,dt = \int_{t_{i-1}}^{t_i} \phi'(t)\,dt = \phi(t_i) - \phi(t_{i-1}) = x_i - x_{i-1}$$
$$\forall\, i = 1, 2 \dots n$$

also $f([x_{i-1}, x_i]) = (f \circ \phi)[t_{i-1}, t_i] \quad \forall\, i = 1 \dots n$

$$L(P, f) = \sum m_i(f)(x_i - x_{i-1})$$
$$= \sum m_i(f) \int_{t_{i-1}}^{t_i} |\phi'(t)|\,dt$$
$$= \sum \int_{t_{i-1}}^{t_i} m_i(f \circ \phi)\, \underbrace{|\phi'(t)|\,dt}$$
$$(f \circ \phi)|\phi'(t)| = \psi(t)$$

on $\phi_i = \phi\big|_{[t_{i-1}, t_i]}$ & $\psi_i = \psi\big|_{[t_{i-1}, t_i]}$
then $|\phi_i'|$ is integrable on $[t_{i-1}, t_i]$ (given)
& $m(f \circ \phi)|\phi_i'| \leq \psi_i$
($\because$ $m_i(f \circ \phi)$ is const $\Rightarrow$ int)
hence
$$L(P, f) = \sum \int_{t_{i-1}}^{t_i} m_i(f \circ \phi)\,|\phi_i'(t)|\,dt$$
$$= \sum L(m_i(f \circ \phi)|\phi_i'|) \leq \sum L(\psi_i)$$

let $\varepsilon > 0$ be given. For each $i = \{1, 2 \dots n\}$



$\exists\, Q_i$ partition of $[t_{i-1}, t_i]$
s.t $L(\psi_i) - \varepsilon/n < L(Q_i, \psi_i)$
let
Q denote the partion of $[\alpha, \beta]$ obtained from the points
$Q_1, Q_2, \dots Q_n$ then

$$\sum L(\psi_i) < \sum L(Q_i, \psi_i) + \varepsilon$$
$$= L(Q, \psi) + \varepsilon \leq L(\psi) + \varepsilon$$

it follows that
$$L(P, f) \leq \sum L(\psi_i) < L(\psi) + \varepsilon$$
$$\Rightarrow L(P, f) \leq L(\psi)$$

$$\Rightarrow L(f) \leq L(\psi)$$
similarly $U(f) \geq U(\psi)$ and as $L(f) = U(f)$
$\Rightarrow L(\psi) = U(\psi) \Rightarrow \psi$ is integrable

also $\int_a^b f(x)dx = \int_\alpha^\beta \psi(t)dt = \int_\alpha^\beta (f\circ\phi)\,|\phi'(t)|dt$

$= \int_{\phi(\alpha)}^{\phi(\beta)} f(t)dt$

as $L(f) = L(\psi)$

The proof is similar for the case of $\phi'(t)<0$ $\forall t\in(\alpha,\beta)$

Defn: (i) let $P=\{x_0,\dots x_n\}$ be a partition of $[a,b]$, let s_i be a point in $[x_{i-1}, x_i]$ $\forall$ $1\le i\le n$. Then

$$S(P,f) := \sum f(s_i)(x_i - x_{i-1})$$

is called riemann sum for f corresponding to P.

(ii) for $P\in P[a,b]$ s.t $P=\{x_0,\dots x_n\}$, we define mesh of P to be

$$\mu(P) = \max\{x_i - x_{i-1} \mid i = 1,2\dots n\}$$

Remark: If $f:[a,b]\to\mathbb{R}$ be integrable and if $\{P_n\}_{n=1}^\infty$ is seq of partition of $[a,b]$ s.t $\mu(P_n)\to 0$

then $L(P_n,f)\to\int_a^b f(x)dx$

& $U(P_n,f)\to\int_a^b f(x)dx$

moreover, if $S(P_n,f)$ is any riemann sum for f corresponding to P_n, then

$$S(P_n,f)\to\int_a^b f(x)dx$$

Example: T.S.t $\sum_{i=1}^{n} \frac{1}{n+i-1} \to \log_e(2)$

$f(x) = \frac{1}{x}$ is integrable over $[1,2]$

let $P_n = \{1, 1+\frac{1}{n}, 1+\frac{2}{n}, \dots 2\}$ be a partition of $[1,2]$

left endpoints $s_{n,i} = 1+\left(\frac{i-1}{n}\right)$ of $[1,2]$

$\mu(P_n) = 1/n \to 0$

$$S(P_n,f) = \sum \frac{1}{1+\frac{(i-1)}{n}}\left(\frac{i}{n} - \frac{i-1}{n}\right) = \sum\frac{1}{n+i-1} \to \int_0^1 \frac{1}{1+x}dx$$

$$= \int_1^2 \frac{1}{x}dx$$

by FTC $\int_a^b \frac{1}{x}dx = \log(b) - \log(a)$

$= \log_e(2)$

23rd oct :

Recall : $S(P_n, f) = \sum_{i=1}^{n} \frac{1}{1+\frac{(i-1)}{n}} \left(\frac{i}{n} - \frac{i-1}{n}\right)$

$$\sum_{i=1}^{n} \frac{1}{n+i-1} \longrightarrow \int_1^2 \frac{1}{x}\,dx = (\log x)_1^2 = \log_e(2)$$

Improper integral of type I :

let $a \in \mathbb{R}$ and f be integrable on $[a, x]$ $\forall x \geq a$. we say an improper integral $\int_a^\infty f(t)dt$ is 'convergent' if $\lim_{x\to\infty} \int_a^x f(t)dt$ exist limit is denoted by $\int_a^\infty f(t)dt$ (similarly $\int_{-\infty}^a f(t)dt$)

Ex : for $a>0$, $\int_a^\infty \frac{dx}{x^p}$ conv iff $p>1$

as let $p \neq 1$ then $\int_a^t \frac{dx}{x^p} = \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - \frac{1}{a^{p-1}}\right]$

if $p>1$ $\lim_{t\to\infty} \int_a^t \frac{dx}{x^p} = \frac{1}{(p-1)a^{p-1}}$

if $p<1$ $\lim_{t\to\infty} \int_a^t \frac{dx}{x^p} \to \infty$ (same for $p=1$ as $\log t \to \infty$)

Cauchy criterion : $\int_a^\infty f(t)dt$ is convergent iff $\forall \varepsilon>0$, $\exists x_0 \in (a,\infty)$ s.t

$\left|\int_x^y f(t)dt\right| < \varepsilon$, $\forall y \geq x \geq x_0$

comparison test : let $\exists x_0 \geq a$ & $k>0$ s.t $\forall x \geq x_0$, $|f(x)| \leq k|g(x)|$

if $\int_a^\infty g(x)\,dx$ is convergent, then $\int_a^\infty |f(x)|dx$ is convg

& $\int_a^\infty |f(x)|\,dx \leq \int_a^\infty |g(x)|\,dx$

limit comparison test : let $g(x) \neq 0$ on $[a,\infty)$. if $\lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| = l$ where l is non-zero finite number then $\int_a^\infty |f(x)|dx$ & $\int_a^\infty |g(x)|dx$ behave alike

$\int f$ is convg $\Leftrightarrow \int g$ is convg

$\int f$ is divg $\Leftrightarrow \int g$ is divg

Improper integral of type 2 : let $a, b \in \mathbb{R}$ with $a<b$. let $f : [a,b) \longrightarrow \mathbb{R}$ be s.t f is unbounded on $[a, b)$ but integrable on $[a, x]$ $\forall x \in [a,b)$

$\int_a^b f(t)dt$ is 'conv' if $\lim_{x\to b-} \int_a^x f(t)dt$ exist. The limit is denoted by $\int_a^b f(t)dt$. Here $\lim_{t\to b^-} |f(t)| = \infty$

(similar notion for $\int_a^b f(t)\,dt$ for $\lim_{t\to a+} |f(t)| = \infty$)

Example: let $a, b \in \mathbb{R}$ $\int_a^b \frac{dx}{(b-x)^p}$ conv iff $p<1$

case I > $(p<1)$

then $\int_a^t \frac{dx}{(b-x)^p} = \left| \frac{(b-x)^{1-p}}{(1-p)} (-1) \right|_a^t$

$= \frac{-1}{(1-p)} (b-t)^{1-p} + \frac{1}{1-p} (b-a)^{1-p}$

for $t \to b^-$ we get $\to \frac{1}{1-p} (b-a)^{1-p}$

for **case II** > $p>1$ $t \to b^- \to \pm\infty$

case III > $p=1$ $\int_a^x \frac{dt}{(b-x)} = \left| \frac{\log(b-x)}{-1} \right|_a^x$

$= -\underbrace{\log(b-x)}_{\to\infty} + \log(b-a)$

$\therefore$ for $p<1 \Leftrightarrow \int_a^b \frac{1}{(b-a)^p} dx$ converges

Limit comparision test: Suppose $\int_a^t |f(x)| dx$ and $\int_a^t |g(x)| dx$ both exist for all $a \leq t < b$ and suppose

$$\lim_{t \to b^-} \left| \frac{f(t)}{g(t)} \right| = \ell$$ where ℓ is a non-zero finite number,

then $\int_a^b |f(x)| dx$ and $\int_a^b |g(x)| dx$ behave alike, i.e both converges or both diverges. (assumed that $g(x) \neq 0$ on $[a, b)$)

Integral test: Assume $f: [1, \infty) \to \mathbb{R}$ be s.t $f(x) \geq 0$ and f is decreasing function. then

$$\int_1^\infty f(x) dx \text{ converge iff } \sum_{n=1}^\infty f(n) \text{ converges}$$

Exe: $\sum_{n=2}^\infty \frac{1}{n(\log n)^p}$

$$\int_2^\infty \frac{dx}{x(\log x)^p} = \lim_{t \to \infty} \int_2^t \frac{dx}{x(\log x)^p} = \lim_{t \to \infty} \int_{\log 2}^{\log t} \frac{du}{u^p}$$

as $t \to \infty$
$\log t \to \infty$

then $\int_{\log 2}^{\log t} \frac{du}{u^p}$ conv iff $\sum_{n=2}^\infty \frac{1}{n(\log n)^p}$ conv

$\Rightarrow$ $\sum \frac{1}{u^p}$ conv $\Leftrightarrow \sum \frac{1}{n(\log n)^p}$

Beta function: let $p, q \in \mathbb{R}$ & $f: (0,1) \to \mathbb{R}$ be a function defined by $f(t) = t^{p-1} (1-t)^{q-1}$

improper integrals:

$\int_0^{1/2} f(t) dt$ and $\int_{1/2}^1 f(t) dt$. If $p \geq 1$ then f is bounded on $(0, \frac{1}{2}]$

$f(0) = 0$ for $p>1$ f is conti $\Rightarrow f$ is integrable on $[0, \frac{1}{2}]$
$f(0) = 1$ for $p=1$

for $p=1$ $f(t)=(1-t)^{q-1}$ $\therefore$ it is cont
$\therefore f$ is integrable on $[0,\frac{1}{2}]$

suppose $p<1$ and let
$g(t)=\frac{1}{t^{1-p}}$ for $t\in(0,\frac{1}{2}]$

then
$$\lim_{t\to 0^+}\frac{f(t)}{g(t)}=\lim(1-t)^{q-1}=1$$

as $\int_0^{1/2} g(t)\,dt$ is conv iff $1-p<1 \Rightarrow p>0$

$\therefore \int_0^{1/2} f(t)\,dt$ is conv iff $p>0$

now if $q\geq 1$ then f is bounded on $[\frac{1}{2},1)$
$f(1)=0$ then f is cont
$\Rightarrow$ integrable on $[\frac{1}{2},1]$

if $q<1$ then for $x\in[\frac{1}{2},1)$ let $y=1-x$ then $y\in(0,\frac{1}{2}]$

$$\int_{1/2}^{x} f(t)\,dt=-\int_{1/2}^{y}(1-u)^{p-1}u^{q-1}\,du$$
$$=\int_y^{1/2}u^{q-1}(1-u)^{p-1}\,du$$

using similar result, we get for $q>0 \Leftrightarrow \int_{1/2}^{1} f(t)\,dt$ is conv

$\underbrace{\beta(p,q)}_{\text{Beta function}}=\int_0^1 t^{p-1}(1-t)^{q-1}\,dt$ for $p>0,\ q>0$

gamma function : $\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}\,dt$ for $s>0$

Note : ① $\Gamma(n+1)=n!$
② $\beta(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ for $p>0,\ q>0$

25^{th} oct :

7. Sequence and series of functions :

Def^n : The series $\sum_{k=0}^{\infty} c_k x^k$ where $x \in \mathbb{R}$ is called power series and real numbers $c_0, c_1, \dots$ are called coeffiets

More generally if $a \in \mathbb{R}$, then the series $\sum_{k=0}^{\infty} c_k (x-a)^k$ where $x \in \mathbb{R}$ is called power series around a. The treatment of such a series can be reduced to a power series around 0 by letting $\tilde{x} = x - a$

lemma 7.1 (Abel's lemma) let x_0 and $c_i \in \mathbb{R}$. If the set $\{c_k x_0^k \mid k \in \mathbb{N}\}$ is bounded, then $\sum_{k=0}^{\infty} c_k x^k$ is absolutely convergent $\forall x \in \mathbb{R}$ with $|x| < |x_0|$. In particular if $\sum_{k=0}^{\infty} c_k x_0^k$ is converged, then $\sum_{k=0}^{\infty} c_k x^k$ is ab. conv $\forall x \in \mathbb{R}$ with $|x| < |x_0|$

proof : if $x_0 = 0$, then the lemma clearly holds, suppose $x_0 \neq 0$. let $\alpha \in \mathbb{R}$ be s.t $|c_k x_0^k| \leq \alpha \quad \forall k \in \mathbb{N}$

suppose $x \in \mathbb{R}$ s.t $|x| < |x_0|$

$$\beta = \frac{|x|}{|x_0|}$$

then $|c_k x^k| = |c_k x_0^k| \beta^k \leq \alpha \beta^k \quad \forall k \in \mathbb{N}$ as $|\beta| < 1$

as $\sum \alpha \beta^k$ is conv

$\Rightarrow \sum |c_k x^k|$ is conv

$\Rightarrow \sum c_k x^k$ is ab. conv

now if $\sum c_k x_0^k$ is conv then

$c_k x_0^k \longrightarrow 0$

$\Rightarrow \{c_k x_0^k \mid k \in \mathbb{N}\}$ is bounded

$\Rightarrow \sum c_k x^k$ is abs conv.

proposition 7.2 : A power series $\sum_{k=0}^{\infty} c_k x^k$ is either absolutely convergent $\forall x \in \mathbb{R}$ or $\exists$ a unique $r \geq 0$ s.t the series is absolutely conv $\forall x \in \mathbb{R}$ with $|x| < r$ & diverges $\forall x \in \mathbb{R}$ with $|x| > r$

proof : let $E = \{|x| \mid x \in \mathbb{R}$ & $\sum c_k x^k$ is conv$\}$

then $0 \in E$. If E is not-bounded then given $x \in \mathbb{R}$, we find $x_0 \in E$ s.t $|x| \leq |x_0|$ and so $\sum c_k x^k$ is absolutely converget by lemma 7.1. Hence in this case $\sum c_k x^k$ is ab. conv $\forall x \in \mathbb{R}$

(Radius of convergence $= r = \infty$)

if E is bounded above then $\exists$ a $\sup E = r$. if $x \in \mathbb{R}$ & $|x| < r$ then by def of sup, $\exists |x_0| \in E$ s.t $|x| < |x_0|$ and so by lemma 7.1 we conclude that $\sum c_k x^k$ is ab. convergent. If $x \in \mathbb{R}$ & $|x| > r$ then by defination of E, $\sum c_k x^k$ is divergent.

$\therefore$ There exist a unique r claimed in the proposition

Def^n : For a power series $\sum c_k x^k$, the radius of convergence is defined as

(i) ∞ if $\sum c_k x^k$ conv $\forall x \in \mathbb{R}$

(ii) unique $r \geq 0$ s.t $\sum c_k x^k$ is ab conv for $|x| < r$ & diverges $|x| > r$

The interval $(-r, r)$ is called int of conv

Example: $\sum x^n$ conv if $|x|<1$ & div $|x|\geq 1 \Rightarrow r=1$

Prop 7.3: let $\sum c_k x^k$ have a radius of conv r.

(i) If $\{|c_k|^{1/k}\}_{k=0}^{\infty}$ is unbounded then $r=0$

(ii) If $\{|c_k|^{1/k}\}_{k=0}^{\infty}$ is bounded $r=\infty$

when $\overline{\lim}\sqrt[n]{|c_n|}=0$ and $r=\dfrac{1}{\lim\sup\sqrt[n]{|c_n|}}$

Proof: $\{|c_k|^{1/k}\}_{k=0}^{\infty}$ is unbounded. Let $x\in\mathbb{R}$ s.t $x\neq 0$. Then there are inf many $k\in\mathbb{N}$ s.t $|c_k|^{1/k}\geq\frac{1}{|x|}$ i.e $|c_k x^k|>1$ so $\sum c_k x^k$ is div

$\therefore$ $r=0$

Suppose $\{|c_k|^{1/k}\}_{k=0}^{\infty}$ is bounded. Then $\overline{\lim}|c_n|^{1/n}<\infty$

let $z\in\mathbb{R}$ & $a_n=c_n z^n$ $\forall n\in\mathbb{N}$. so

$$\overline{\lim_{n\to\infty}}|a_n|^{1/n}=|z|\overline{\lim_{n\to\infty}}|c_n|^{1/n}$$

by root test

$$\overline{\lim_{n\to\infty}}|a_n|^{1/n} \text{ conv if } \overline{\lim}|a_n|^{\frac{1}{n}}<1$$

$\Rightarrow \overline{\lim}|c_n|^{1/n}|z|<1$

& div for

$|z|\overline{\lim}|c_n|^{\frac{1}{n}}>1$

when $|z|<\dfrac{1}{\overline{\lim}|c_n|^{1/n}}$

$\therefore$ when $\overline{\lim}|c_n|^{1/n}=0$

$r=\infty$

else $r=\dfrac{1}{\overline{\lim}|c_n|^{1/n}}$

Prop 7.4: let r be radius of conv of $\sum c_k x^k$.

(i) If $|c_{k+1}|/|c_k|\to\infty$ as $k\to\infty$ then $r=0$

(ii) If $\left\{\left|\frac{c_{n+1}}{c_n}\right| : n\in\mathbb{N}\right\}$ is bounded then

$r=\infty$ for $\lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|}=0$

& $r=\dfrac{1}{\lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|}}$ otherwise

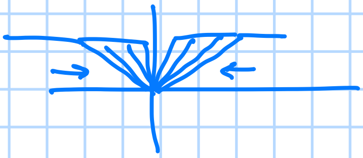
Proof: Similar to Prop 7.3

Defn: let E be a set and Y be a metric space. Consider functions $f_n: E\to Y$ $\forall n\in\mathbb{N}$. we say the seq $\{f_n\}_{n=1}^{\infty}$ conv pointwise on E if $\exists f: E\to Y$ s.t $f_n(p)\to f(p)$ $\forall p\in E$

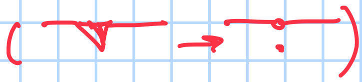
Clearly f is unique. Here we deal with $Y=\mathbb{R}$

Example: (i) let $E = [-1,1]$ and define $f_n : E \longrightarrow \mathbb{R}$ define $f_n : E \longrightarrow \mathbb{R}$ by

$$f_n(x) = \begin{cases} n|x| \; ; & 0 \le |x| \le 1/n \\ 1 \; ; & 1/n < |x| < 1 \end{cases}$$



$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1, & 0 < |x| \le 1 \end{cases}$$



28th Oct :

Power series : $\sum c_n x^n$; $c_1, c_2, \ldots \in \mathbb{R}$, $x \in \mathbb{R}$

$$f_k(x) = \sum_{n=1}^{k} c_n x^n$$

defn : E be a set and Y be a metric space. Consider function $f_n : E \to Y$ $\forall n \in \mathbb{N}$. We say seq $\{f_n\}_{n=1}^{\infty}$ conv pointwise on E. If

$\exists f : E \to Y$ s.t.

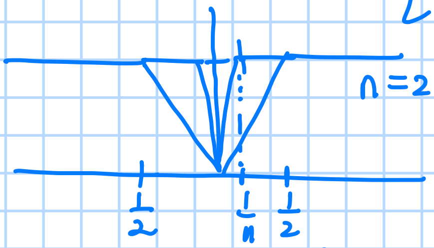
$f_n(P) \longrightarrow f(P)$ $\forall P \in E$

f is unique

Example :

$E = [-1,1]$ $f_n : E \to \mathbb{R}$

$$f_n(x) = \begin{cases} n|x| ; & 0 \le |x| \le 1/n \\ 1 ; & 1/n < |x| \le 1 \end{cases}$$



as $n \to \infty$

$$f_n(x) \longrightarrow f(x) = \begin{cases} 1 ; & x \in [-1,1] \setminus \{0\} \\ 0 ; & x = 0 \end{cases}$$

f_n is cont $\not\Rightarrow$ f is cont

Note : $f_n : E \to Y$

$d(f(x), f_n(x)) < \varepsilon$

Because of this, we use metric spaces

If f_n are cont (at c) then is f continuous (at c) ?

$\lim_{x \to c} f_n(x) = f_n(c)$ (given)

check: $\lim_{x \to c} f(x) = f(c)$

as : $\lim_{x \to c} f_n(x) = f_n(c)$

as f_n is cont

$f_n(\lim_{x \to c} x) = f_n(c)$

& $\lim_{n \to \infty} f_n(x) = f(x)$

$$\lim_{x \to c} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} f_n(\lim_{x \to c} x)$$

$= \lim_{n \to \infty} \lim_{x \to c} f_n(x)$ $\longrightarrow$ The question is can we really do this (for pointwise, we cannot)

or Pointwise is a weak convergence

$\Downarrow$

Stronger conv exist. (this is possible)

Ex : $E = (-1,1)$ $f_n : E \to \mathbb{R}$

$$f_n(x) = \sqrt{x^2 + \frac{1}{(n)^2}}$$

$f_n \to f$ on E pointwise where $f(x) = |x|$ $\forall x \in E$

f_n at 0

$$f_n'(0) = \lim_{h \to 0} \frac{f_n(h) - f_n(0)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{h^2 + 1/n^2} - 1/n}{h}$$

$= 0$

Note : f is cont but not diff on $|x|$ (at 0)

Note: Pointwise conv: f_n is diff $\not\Rightarrow f$ is diff
f_n is cont $\not\Rightarrow f$ is cont

Ex: $E = [0,1]$ $f_n : E \to \mathbb{R}$

$$f_n(x) = \underbrace{n^3 x e^{-nx}}_{\text{Riemann integrable}}$$

$f_n \to f$ when $n \to \infty$
or $f = 0$

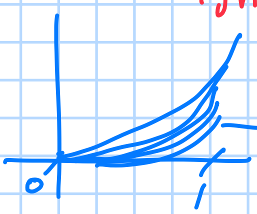
then
$$\int_0^1 f_n(x)\,dx = \int_0^1 n^3 x e^{-nx}\,dx$$
$$= n^3 \int_0^1 x e^{-nx}\,dx$$
$$= n^3 \left[x \frac{e^{-nx}}{-n} + \int_0^1 \frac{e^{-nx}}{+n}\,dx \right]$$
$$= n^3 \left[x\frac{e^{-nx}}{-n} + \frac{e^{-nx}}{-n^2} \right]_0^1$$
$$= \left[-n^2 x e^{-nx} - n e^{-nx} \right]_0^1$$
$$= \left[n^2 (1) e^{-n} - n e^{-n} \right] - [0 - n]$$
$$= \underbrace{-n^2 e^{-n}}_{\to 0} \underbrace{- n e^{-n}}_{\to 0} \underbrace{+ n}_{\to \infty}$$

but
$$\int_0^1 f(x)\,dx = 0$$

Defn: let E be a set and consider function $f_n : E \to \mathbb{R}$ $\forall n \in \mathbb{N}$. A seq $\{f_n\}_{n=1}^{\infty}$ of functions conv uniformly on E if $\exists$ a function $f : E \to \mathbb{R}$ s.t for $\varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ we have

$$|f_n(p) - f(p)| < \varepsilon \quad \forall p \in E,\ n \geq n_0$$

Eg:



$\to f_n(x) = x^n$ as $n \to \infty$
$f_n \to 0$ (Pointwise)

let $0 < r < 1$
$E = [-r, r]$
$f_n(x) = x^n \quad \forall x \in E$

$f_n \to f$ pointwise
for $f = 0$

let $a_n := r^n$ $\forall n \in \mathbb{N}$ then $a_n \to 0$
then
it is uniform conv for $[-r, r]$
as $|f_n(x) - 0| \leq \underset{\downarrow 0}{a_n} \to 0$
$\forall x \in [-r, r]$

Note: Pointwise $\not\Rightarrow$ uniform conv

eg: $E = (0,1]$ $\quad f_n(x) = \frac{1}{nx+1} \longrightarrow 0 = f(x)$ as $n \to \infty$

$\therefore f_n \xrightarrow{} f$ (Pointwise)

let $\varepsilon = 1/4$ $\quad \left| f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right) \right| = \frac{1}{2} > \varepsilon$

for any $n \to \infty$ $\quad \left| f_n\left(\frac{1}{n}\right) \right| = \frac{1}{2}$

$\therefore$ we cannot find n_0 s.t

$|f_n(x) - f(x)| < \varepsilon$
$\forall n \geq n_0$ & $x \in E$

$\therefore f_n$ are not uniformly conv

prop 7.5: let $\{f_n\}_{n=1}^{\infty}$ be a seq of function on E. Then $\{f_n\}_{n=1}^{\infty}$ is uniformly conv on E $\Leftrightarrow$ $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t $\forall m, n \geq n_0$
($f_n : E \to \mathbb{R}$) $\quad |f_m(x) - f_n(x)| < \varepsilon \ \forall x \in E$

proof: ($\Rightarrow$) $f_n \to f$ uniformly and $\varepsilon > 0$. There $\exists n_0 \in \mathbb{N}$ satisfying

$|f_n(x) - f(x)| < \varepsilon/2 \quad \forall n \geq n_0, \ x \in E$

$\forall n, m \geq n_0$ & $x \in E$

$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

($\Leftarrow$) conversely, let the cauchy condition be hold

fix $x \in E$, for this

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t
$|f_n(x) - f_m(x)| < \varepsilon \quad \forall n, m \geq n_0$

as $\mathbb{R}$ is complete
$\Rightarrow f_n \to f$ for x point

if we define this f for every point $x \in E$

fix $n \geq n_0$ & let $m \to \infty$
then $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t $\forall n \geq n_0$,
$|f(x) - f_n(x)| < \varepsilon \quad \forall x \in E$
(uniform convergence)

prop 7.6: let $\{f_n\}_{n=1}^{\infty}$ be a seq of real-valued funtions on $[a,b]$. If $f_n \to f$ uniformly on $[a,b]$ & each f_n is Riemann integrable on $[a,b]$ then f is Riemann integrable on $[a,b]$

& $\int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx$

proof: since $f_n \to f$ uniformly, $\exists n_0 \in \mathbb{N}$ s.t $|f_n(x) - f(x)| < 1 \quad \forall n \geq n_0$ & $\forall x \in [a,b]$

$|f_n(x)| \leq \alpha_n \quad \forall x \in [a,b]$

$\Rightarrow |f(x)| \leq |f_{n_0}(x) - f(x)| + |f_{n_0}(x)| < 1 + \alpha_{n_0}$

or $f(x)$ is bounded

As f is bounded, here $\beta_n := \sup\{ |(f_n - f)(x)| \mid x \in E \}$. Then $\beta_n \to 0$ ($\because$ uniform cont)

we have $-\beta_n \leq -f_n(x) + f(x) \leq \beta_n$

$\Rightarrow f_n(x) - \beta_n \leq f(x) \leq f_n(x) + \beta_n \quad \forall\ 0 \leq k \leq s$

$P = \{x_0, \dots x_s\}$ of $P[a,b]$

$m_k(f_n) - \beta_n \leq m_k(f)$

$M_k(f) \leq M_k(f_n) + \beta_n \quad \forall\ 0 \leq k \leq s-1$

$\Rightarrow \sum m_k(f_n)(\Delta x_i) - \beta_n(b-a) \leq \sum m_k(f)(\Delta x_i) \leq L(f)$

$\Rightarrow L(f_n) - \beta_n(b-a) \leq L(f)$

Similarly $U(f) \leq U(f_n) + \beta_n(b-a)$

$\Rightarrow L(f_n) \leq U(f_n) + 2\beta_n(b-a)$

as $\beta_n \to 0$ for $n \to \infty$

$\Rightarrow L(f_n) = U(f_n) = U(f) = L(f)$

$\Rightarrow \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$

Note: integrable in uniform cont.

29th oct:

Prop 7.7: let $\{f_n\}_{n=1}^{\infty}$ be seq of real valued functions defined on a metric space E. If $f_n \to f$ uniformly on E and each f_n is cont on E, then f is cont on E.

$\left(\{f_n\}_{n=1}^{\infty},\ \lim_{x\to c} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to c} f_n(x), \text{ we can interchange lim} \right)$

proof: let $\varepsilon > 0$, as $f_n \to f$ uniformly, $\exists\, n_0 \in \mathbb{N}$ s.t

$|f_{n_0}(x) - f(x)| < \varepsilon/3 \ \forall x \in E$

let $x_0 \in E$

as f_{n_0} is cont at x_0, $\exists\, \delta > 0$

s.t $d(x, x_0) < \delta \Rightarrow |f_{n_0}(x) - f_{n_0}(x_0)| < \varepsilon/3$

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)|$$

$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$

for $d(x, x_0) < \delta$

$|f(x) - f(x_0)| < \varepsilon$

$\therefore$ f is cont at x_0

Prop 7.8: let $\{f_n\}_{n=1}^{\infty}$ be a seq of real valued functions defined on $[a,b]$. If $\{f_n\}_{n=1}^{\infty}$ converges at a point, each f_n is cont. diff on $[a,b]$ and $\{f'_n\}$ convgs uniformly on $[a,b]$, then $\exists$ a continuously diff function $f: [a,b] \to \mathbb{R}$ s.t

$f'_n \to f'$ on $[a,b]$

& $f_n \to f$ uniformly

proof: let $x_0 \in [a,b]$

& $c_0 \in \mathbb{R}$ s.t $f_n(x_0) \to c_0$

each f_n is cont. diff

& $\{f'_n\}$ convg uniformly to g

$\forall \varepsilon > 0$, $\exists\, n_1$ & n_2 s.t

$|f_n(x_0) - c| < \varepsilon \ \forall n \geq n_1$

& $|f'_n(x) - g(x)| < \varepsilon \ \forall n \geq n_2$

By Prop 7.7 g is cont on $[a,b]$

By FTC Part I we get for $n \in \mathbb{N}$

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f'_n(t)\,dt$$

define $f: [a,b] \to \mathbb{R}$ by

$$f(x) = c_0 + \int_{x_0}^{x} g(t)\,dt$$

By FTC part II f is diff and $f' = g$ on $[a,b]$. Thus f is cont diff on $[a,b]$ & $f'_n \to f'$ uniformly.

further for $n \geq n_0 = \max\{n_1, n_2\}$ & $x \in [a,b]$

$$|f_n(x) - f(x)| \leq |f_n(x_0) - c_0| + \left|\int_{x_0}^{x} f'_n(t) - g(t)\,dt\right|$$

$$\leq |f_n(x_0) - c_0| + |x - x_0| \sup_{t \in [a,b]} |f'_n(t) - g(t)|$$

$< \varepsilon + (b-a)\varepsilon$

$\therefore f_n \to f$ uniformly on $[a,b]$

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f'_n(t)\,dt$$

$f(x) \qquad c_0 \qquad g$

$f: [a,b] \longrightarrow \mathbb{R}$

$$f(x) = f(x_0) + \int_{x_0}^{x} g(t)\,dt$$

$f(x_0) = c_0$

$$f(x) = c_0 + \int_{x_0} g(t)\,dt$$

$f' = g$

$$\left| f_n(x) - \left(c_0 + \int_{x_0}^{x} g(t)\,dt \right) \right|$$

$$\leq \left| f_n(x_0) - f_n(x_0) + f_n(x) - c_0 - \int_{x_0}^{x} g(t)\,dt \right|$$

$$\leq |f_n(x_0) - c_0| + \left| \int_{x_0}^{x} f'_n(t)\,dt - \int_{x_0}^{x} g(t)\,dt \right| \leq \varepsilon + (b-a)(\varepsilon)$$

Prop 7.9 : (Weierstrass M-test)

let $\{f_k\}_{k=1}^{\infty}$ be a seq of real-valued function defined on a set E. suppose $\exists$ a seq $\{M_k\}_{k=1}^{\infty}$ in $\mathbb{R}$ s.t $|f_k(x)| \leq M_k$ $\forall k \in \mathbb{N}$ and $x \in E$. If $\sum_{k=1}^{\infty} M_k$ is convg, then $\sum_{k=1} f_k$ converges uniformly & absolutely on E.

proof : $\left|\sum_{k=n}^{m} f_k(x)\right| \leq \sum_{k=n}^{m} |f_k(x)| \leq \sum_{k=n}^{m} M_k \quad \forall\, m \geq n$

$|f_k(x)| \leq M_k$

$$h_n(x) = \sum_{k=1}^{n} f_k(x)$$

$\left|\sum f_k(x)\right| \leq \sum |f_k(x)| \leq \sum M_k$

$$|h_m(x) - h_n(x)| = \left|\sum_{k=n+1}^{m} f_k(x)\right| \quad \forall\, m \geq n$$

$$\leq \sum_{k=n+1}^{m} |f_k(x)|$$

$$\leq \sum_{k=n+1}^{m} M_k$$

as $\{M_k\}$ convg $\Rightarrow$ $|h_m(x) - h_n(x)| < \varepsilon$ can be made

$\Rightarrow$ $|\sum f_k(x)|$ convgs

&

$\sum |f_k(x)|$ convgs

(Cauchy criterion is used here)

30th oct :

functions defined using power series on an open set are called analytic functions.

① open set
② using power series

Theorm 7.10 : suppose the series $\sum_{n=0}^{\infty} c_n x^n$ which converges for $|x|<R$ and define

$f(x)=\sum_{n=0}^{\infty} c_n x^n$ for $|x|<R$, then the series conv uniformly on $[-R+\varepsilon, R-\varepsilon]$ $\forall \varepsilon>0$

The function f is cont and diff on $(-R,R)$ and

$$f'(x)=\sum_{n=1}^{\infty} n c_n x^{n-1} \text{ for } |x|<R$$

proof : let $\varepsilon>0$ be given.

For $|x|\leq R-\varepsilon$ we have $|c_n x^n| \leq |c_n (R-\varepsilon)^n|$

and since

$\sum |c_n (R-\varepsilon)^n|$ converges (property of power series)

by weierstrass M-test, the series

$\sum_{n=0}^{\infty} c_n x^n$ converges uniformly on $[-R+\varepsilon, R+\varepsilon]$

(weierstrass M-test)

since $\sqrt[n]{n} \to 1$ as $n \to \infty$ we have

$$\overline{\lim} \sqrt[n]{n|c_n|} = \overline{\lim} \sqrt[n]{n} \sqrt[n]{|c_n|} = \overline{\lim} \sqrt[n]{|c_n|}$$

hence

$\sum n c_n x^{n-1}$ and $\sum c_n x^n$ have same interval of convergence.

As $\sum n c_n x^{n-1}$ is a power series it converges uniformly on $[-R+\varepsilon, R-\varepsilon]$ $\forall \varepsilon>0$. By theorm 7.8 $\sum n c_n x^{n-1} = f_n'(x)$ holds if $|x| \leq R-\varepsilon$
but for $x\in\mathbb{R}$ we have $|x|<R \Rightarrow \exists \varepsilon>0$ s.t $|x|<R-\varepsilon$

$\therefore f_n'(x) = \sum n c_n x^{n-1}$ holds for $|x|<R$

$f_n' \to f'$ $\left\{ \lim_{n\to\infty} f_n' = \left(\lim_{n\to\infty} f_n\right)' \right.$

$$f' = \lim_{n\to\infty} \left(\sum c_k x^k\right)'$$
$$= \lim_{n\to\infty} \left(\sum k c_k x^{k-1}\right)$$ Because f_n' is uniform conv

corollary 7.11 : Under the hypothesis of thm 7.10, f has derivatives of all orders in $(-R,R)$ which is given by

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) c_n x^{n-k}$$

in particular, $f^{(k)}(0) = k!\, c_k$ $\forall k\in\mathbb{N}\cup\{0\}$

proof : Applying $f'(x)=\sum k c_k x^{k-1}$ repeatedly & putting $x=0$ we get the above.

$f^{(n)}(0) = n!\, c_n \leftarrow n^{th}$ derivative of 0

Any power series (given inside $(-R,R)$)

very similar to taylor series terms

Note: for a smooth function (inf. many diff.) whose remainder $\to 0$ for taylor series gives series (polynomial)

Exponential function:

$$E(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$\left(E(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)$

as $\lim_{n\to\infty}\frac{(n+1)!}{n!}=\infty$ ratio test shows that the series conv $\forall x$.

$$E(x)E(y)=\left(\sum_n\frac{x^n}{n!}\right)\left(\sum_m\frac{y^m}{m!}\right)$$

$$=\sum_n\sum_{k=0}^{n}\frac{x^k y^{n-k}}{k!(n-k)!}$$

$\left(\sum_n\frac{x^n}{n!}\sum_m\frac{y^m}{m!}\right)=\left(\sum_n^{\infty}\sum_k^{n}\frac{x^k y^{n-k}}{k!\,(n-k)!}\right)$

$$=\sum_n\frac{1}{n!}\left[\sum_k\frac{n!}{k!(n-k)!}x^k y^{n-k}\right]$$

$$=\sum_n\frac{1}{n!}\left((x+y)^n\right)$$

$$=E(x+y)$$

Note: $E(x)E(y)=E(x+y)$ is important property of $E(x)$

$\therefore E(x)E(-x)=E(0)=1$

thus $E(x)\neq 0\ \forall x\in\mathbb{R}$

as $E(x)>0$ for $x>0$

$\therefore E(x)>0\ \forall x\in\mathbb{R}$

$E(x)\to\infty$ as $x\to\infty$

By $E(x)=\sum\frac{x^n}{n!}$ and $\lim_{n\to\infty}\frac{(n+1)!}{n!}\to\infty$

then $E(x)\to 0$ as $x\to-\infty$

as $E(x)E(-x)=1$

as $x\to\infty$

$-x\to-\infty$

$E(-x)=\frac{1}{E(x)}\to 0$

$\therefore E(x)\to 0$ & $x\to-\infty$

Note: $0<x<y\Rightarrow E(x)<E(y)$

as $0<x<y$

$\Rightarrow x^n<y^n$

$\Rightarrow \sum x^n<\sum y^n$

$\Rightarrow E(x)<E(y)$

$\left(\frac{x}{y}<1\Rightarrow\frac{x^n}{y^n}<1\Rightarrow x^n<y^n\right)$

By the fact $E(-y)=\frac{1}{E(y)}$

we have

$E(x)<E(y)\Rightarrow E(-y)<E(-x)$

$\therefore E$ is strictly increasing

further $\lim_{h \to 0} \frac{E(x+h) - E(x)}{h} = E(x) \lim_{h \to 0} \frac{E(h)-1}{h} = E(x) \lim_{h \to 0} \frac{1+\cdots-1}{h}$

$$= E(x) E'(0)$$
$$= E(x)$$

Note: $E(x) = E'(x)$

Recall $E(1) = \sum_n \frac{1}{n!} = e$. By the fact $E(2) = E(1) \cdot E(1)$ we get $E(n) = e^n$

Note: $E(n) = e^n$

if $p = n/m$ where $n, m \in \mathbb{N}$ then

$$(E(p))^m = E(mp) = E(n) = e^n$$
$$\Rightarrow E(p) = e^{n/m}$$

Note: $E\left(\frac{n}{m}\right) = e^{\frac{n}{m}} \quad \forall n/m \in \mathbb{Q}$

And now $E(-p) = e^{-n/m}$ so similar thing

define $e^x = \sup_{p<x,\, p \in \mathbb{Q}} e^p$ for $x \in \mathbb{R}$

$$= E\left(\sup_{p<x,\, p\in\mathbb{Q}} p\right) \quad (\because E \text{ is cont } \& \text{ inc})$$

$e^x = E(x) \ \forall x \in \mathbb{R}$

Note: $e^x = E(x) \ \forall x \in \mathbb{R}$

Remark: $f: I \to \mathbb{R}$ be an injective cont function, I is an interval. Let c be interior point of I & $f^{-1}: f(I) \to I$ be the inverse function. If I is diff at c & $f'(c) \neq 0$, then f^{-1} is diff at $f(c)$ &

$$(f^{-1})'(f(c)) = \frac{1}{f'(c)}$$

Proof: $x = (f^{-1} f)(x)$

$$1 = (f^{-1})'(f(x)) f'(x)$$

at $c \Rightarrow \frac{1}{f'(c)} = f^{-1\prime}(f(c))$

Logarithm function:

Since E is strict inc on $\mathbb{R}$, it has an inverse L, i.e.

$E(L(y)) = y$ for $y > 0$

or $L(E(x)) = x \ \forall x \in \mathbb{R}$

as E is diff $\Rightarrow L$ is diff

$L'(y) = \frac{1}{E'(L(y))} > 0$ or L is strictly inc

$L'(E(x)) E(x) = 1 \quad y = E(x)$

$L'(y) = \frac{1}{y} \ \forall y > 0$

we denote $L(x)$ as $\log x$.

$\log(1)=0$

$\log(y)=\int_1^y \frac{dx}{x}$

$\log(uv)=\log(u)+\log(v)$

$\log(x)\to\infty$ as $x\to\infty$

$\log(x)\to-\infty$ as $x\to 0$

$L'(y)=\frac{1}{y}\quad \forall y>0$

$L(E(x))=x$

as for $x\to-\infty$

$E(x)\to 0$

$L(y)\to-\infty$

for $y\to 0$

sim $L(y)\to\infty$

for $y\to\infty$

Note: $L(\underbrace{E(x)}_{u}\underbrace{E(y)}_{v}) = L(E(x+y))$

$= x+y$

$= L(\underbrace{E(x)}_{u}) + L(\underbrace{E(y)}_{v})$

or $L(uv)=L(u)+L(v)$

$\Rightarrow L(1)=L(1)+L(1)$

$\Rightarrow L(1)=0$

now $L'(y)=\frac{1}{y}$

$\Rightarrow L(y)=L(1)+\int_1^y \frac{1}{t}\,dt$

$\Rightarrow L(y)=\int_1^y \frac{1}{t}\,dt$ which is our defination for log function

5^{th} Nov :

recap :

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \quad |x| < R, \quad |x|<R$$
$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad |x|<R$$
conv

$\Rightarrow f^{(k)}(0) = k!\, c_k$

$\Rightarrow c_k = \frac{f^{(k)}(0)}{k!} \quad \forall k \in \mathbb{N} \cup \{0\}$

$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ This function $(E(x))$ had properties similar to e^x and then using continuity can be concluded that it is e^x.

where $E(x+y) = E(x)E(y)$

as : $(a_0 + a_1)(b_0 + b_1) = a_0 b_0 + a_0 b_1 + a_1 b_0 + b_0 b_1$

$\sum a_i \quad \sum b_i \quad = \sum c_i$

$$\sum c_i = a_0 b_n + a_1 b_{n-1} + \dots \; a_n b_0$$

gamma function :

claim: $\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx$ conv iff $s > 0$

let $f(x) = x^{s-1} e^{-x}$

$$I_1 = \int_0^1 x^{s-1} e^{-x} dx \qquad I_2 = \int_1^{\infty} x^{s-1} e^{-x} dx$$

If $s \geq 1$, $x^{s-1} e^{-x}$ is cont $\Rightarrow I_1$ is proper / Riemann

$s < 1$, $g(x) = x^{s-1}$ & hence

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{1}{e^x} = 1$$

$\therefore f(x)$ & $g(x)$ behave alike

$$\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x^{1-s}}\,dx = \left| \frac{x^{s-1+1}}{s-1+1} \right|_0^1$$
$$= \left| \frac{x^s}{s} \right|_0^1$$

conv iff $s > 0$

$\therefore$ for $s > 0$, I_1 converges

for I_2, $\exists\, m \in \mathbb{N}$ s.t $s < m$. Hence for $x \geq 1$,
$x^s < x^m$
i.e $x^{s+1} < x^{m+1}$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots \frac{x^{m+1}}{(m+1)!} + \dots$$
$$> \frac{x^{m+1}}{(m+1)!} \quad \text{for } x \geq 1$$

$\therefore$ for $x \geq 1$ $e^x > \frac{x^{s+1}}{(s+1)!}$ i.e $x^{-2}(m+1)!$

$> (x)^{s-1} e^{-x}$

$$x^{s-1} e^{-x} < x^{-2}(m+1)!$$

as $\displaystyle\int_1^\infty \frac{dx}{x^2}\,(m+1)! > \int_1^\infty x^{s-1} e^{-x}\,dx$

$$\Rightarrow (m+1)!\,(1) > \int_1^\infty x^{s-1} e^{-x}\,dx$$

$\therefore \displaystyle\int_1^\infty x^{s-1} e^{-x}\,dx$ converges

$\therefore \Gamma(s) = \displaystyle\int_0^\infty x^{s-1} e^{-x}\,dx$ conv for $s>0$

<u>Prop 7.12</u>: (i) $\Gamma(n+1) = n\,\Gamma(n)$, $\Gamma(1) = 1$ $\forall n \in \mathbb{N}$

(ii) $\Gamma(n+1) = n!$ $\forall n \in \mathbb{N}$

<u>proof</u>:

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx$$

$$= \cancel{\left| x^n \frac{e^{-x}}{-x} \right|_0^\infty} + \int_0^\infty n x^{n-1} \frac{e^{-x}}{+x}\,dx$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

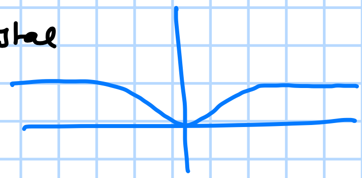
$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1$$

<u>Note</u>: If $f \in C^\infty(I)$ ($f: I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, f has derivative of every order)

If $f \in C^\infty(I)$ on a neighbourhood of point c, then power series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ is called <u>taylor series</u> about c generated by f

Every analytic function is $C^\infty(I)$, from exe 5.4 Apostol

$$f(x) = \begin{cases} e^{-1/x^2} ; & x \neq 0 \\ 0 ; & x = 0 \end{cases}$$



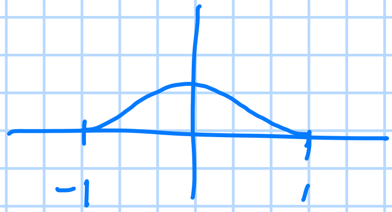
is diff over $\mathbb{R}$ & $f^{(n)}(0) = 0$ $\forall n \in \mathbb{N}$

so $f \in C^\infty(\mathbb{R})$. But the taylor series of f is 0 (radius of conv ∞), but which is <u>not</u> equal to f on any neighbourhood to 0. So f is <u>not</u> analytic (cannot be rep as power series for 0)

Not all $C^\infty(I)$ functions are analytic in nature

<u>Bump function</u>: $\psi: \mathbb{R} \to \mathbb{R}$

$$\psi(x) = \begin{cases} E\left(-\frac{1}{1-x^2}\right) ; & |x| < 1 \\ 0 ; & |x| \geq 1 \end{cases} \qquad \psi \in C^\infty(\mathbb{R})$$



Ψ is a smooth function which has compact support

$f \in C([a,b])$, for $f(x) \geqslant 0 \ \forall x \in [a,b] \Rightarrow f \geqslant 0$

$\hookrightarrow$ cont function set over $[a,b]$

Theorm 7.13: (Korovkin-1) suppose $f_0(x)=1$, $f_1(x)=x$ & $f_2(x)=x^2$ $\forall x \in [a,b]$.

Let $P_n : C([a,b]) \rightarrow C([a,b])$ be linear maps s.t $P_n(f) \geqslant 0$ if $f \geqslant 0 \longrightarrow$ (#)

If $P_n(f_j) \rightarrow f_j$ uniformly $\forall j = 0,1,2$ then $P_n(f) \rightarrow f$ uniformly on $[a,b]$.

$P_n(f_0) \searrow f_0$, $P_n(f_1) \searrow f_1$, $P_n(f_2) \searrow f_2$ } uniformly

then

$P_n(f) \longrightarrow f$ uniformly

proof: If $f \in C([a,b])$ then $f = \operatorname{Re} f + i \operatorname{Im} f$ where

$\operatorname{Re} f, \operatorname{Im} f \in C([a,b]) \rightarrow$ Real valued

Because $P_n(f) = P_n(\operatorname{Re} f) + i P_n(\operatorname{Im} f) \ \forall n \in \mathbb{N}$

$\therefore$ we can prove for real valued.

$f \in C([a,b])$ be real valued

f is bounded

$\exists \alpha \in \mathbb{R}$

s.t $|f(x)| \leqslant \alpha \quad \forall x \in [a,b]$

$\therefore -2\alpha \leqslant f(x) - f(y) \leqslant 2\alpha$ —— (a)

for $\varepsilon > 0$ $\exists \delta > 0$ s.t (As uniform cont)

$x, y \in [a,b]$

$|x - y| < \delta \Rightarrow -\varepsilon < f(x) - f(y) < \varepsilon$ —— (b)

fix $x \in [a,b]$ $f_x(y) = (y-x)^2 \ \forall y \in [a,b]$

for $|x-y| \geqslant \delta \Rightarrow f_x(y) \geqslant \delta^2$

combo (a), (b)

so $\forall y \in [a,b]$

$-\varepsilon - 2\alpha \leqslant f(x) - f(y) \leqslant \varepsilon + 2\alpha$

$\Rightarrow -\varepsilon - \frac{2\alpha}{\delta^2} f_x(y) \leqslant f(x) - f(y) \leqslant \varepsilon + \frac{2\alpha}{\delta^2} f_x(y)$

using (#) and Linearity of P_n we obtain

$$-\varepsilon P_n(f_0) - \frac{2\alpha}{\delta^2} P_n(f_x) \leqslant P_n(f) - f(x) P_n(f_0) \leqslant \varepsilon P_n(f_0) + \frac{2\alpha}{\delta^2} P_n(f_x)$$ —— C

Here $P_n(f_0) \rightarrow f_0$ uniformly so, $\exists m \in \mathbb{N}$ s.t

$|P_n(f_0)(y) - f_0(y)| < \varepsilon/\alpha \ \forall n \geqslant m \ \forall y \in [a,b]$ —— (*)

also $f_x = f_2 - 2x f_1 + x^2 f_0$

$\Rightarrow P_n(f_x) = P_n(f_2) - 2x P_n(f_1) + x^2 P_n(f_0)$

so By aromprious $P_n(f_x)(x) \longrightarrow x^2 - 2x^2 + x^2 = 0$
uniformly on $[a, b]$

$\therefore\ \exists\, n_0 \in \mathbb{N}$ s.t $\forall n \geq n_0$

$$\left| \frac{2a}{\delta^2} P_n(f_x)(x) \right| < \varepsilon/2 \ \&$$

$$\left| P_n(f_0)(x) - 1 \right| < 1/2$$

$$\Rightarrow \quad 1/2 \leq P_n(f_0)(x) \leq 3/2$$

& from ⓒ we get

$$-\varepsilon/2 - \varepsilon/2 \leq P_n(f)(x) - f(x) P_n(f_0)(x) \leq 3\varepsilon/2 + \varepsilon/2 \quad \text{— ⓓ}$$

so for $n \geq \max\{n_0, m\}$, $-3\varepsilon \leq P_n(f) - f \leq 3\varepsilon$ (using ⓓ and ⊛)

$\therefore\ P_n(f) \to f$ uniformly on $[a, b]$.

6^{th} Nov:

Theorm 7.14: (Weierstrass approximation) Every real valued cont function on $[0,1]$ is the uniform limit of a sequence of real-valued polynomial function on $[0,1]$.

proof: nth Bernstein polynomial of a function $f:[0,1]\to\mathbb{R}$ defined by

$$B_n(f)(x)=\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)x^k(1-x)^{n-k}$$

for $c_1, c_2 \in \mathbb{R}$ we have

$$B_n(c_1 f+c_2 g)(x)$$

$$=\sum\binom{n}{k}(c_1 f+c_2 g)\left(\frac{k}{n}\right)x^k(1-x)^{n-k}$$

$$=c\sum\binom{n}{k}f\left(\frac{k}{n}\right)x^k(1-x)^{n-k}+c_2\sum\binom{n}{k}g\left(\frac{k}{n}\right)x^k(1-x)^{n-k}$$

so B_n is linear. since $\binom{n}{k}$, x^k & $(1-x)^{n-k}$ are positive,

$B_n(f)\geq 0$ if $f\geq 0$. let $f_0(x)=1$, $f_1(x)=x$, $f_2(x)=x^2$ $\forall x\in[0,1]$

$$B_n(f_0)(x)=\sum\binom{n}{k}x^k(1-x)^{n-k}$$

$$=(\cancel{x}+1-\cancel{x})^n=1^n=1$$

$$\therefore B_0(f_0)=f_0$$

$$B_n(f_1)(x)=\sum\binom{n}{k}\left(\frac{k}{n}\right)x^k(1-x)^{n-k}$$

$$=\sum\frac{(n-1)!}{(n-k)!\,(k-1)!}\frac{\cancel{(k)}}{\cancel{(k)}}x^k(1-x)^{n-k}$$

$$=x\sum\frac{(n-1)!}{(n-1-k+1)(k-1)!}x^{k-1}(1-x)^{n-1-k+1}$$

$$=x(\cancel{x}+1-\cancel{x})^{n-1}$$

$$=x=f_1(x)$$

$$B_n(f_1)(x)=f_1(x)$$

$$B_n(f_2)(x)=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}\right)^2x^k(1-x)^{n-k}$$

$$=\sum\frac{(n)!}{(n-k)!\,(k)!}\frac{k}{n}\cdot\frac{k}{n}x^k(1-x)^{n-k}$$

$$=x\left[\sum\binom{n-1}{k-1}\left(\frac{k}{n}\right)x^{k-1}(1-x)^{n-1-(k-1)}\right]$$

$$=x\left[\sum\binom{n-1}{k-1}\left(\frac{k-1}{n}\right)x^{k-1}(1-x)^{n-1-(k-1)}\right.$$
$$\left.+\sum\binom{n-1}{k-1}\left(\frac{1}{n}\right)x^{k-1}(1-x)^{n-1-(k-1)}\right]$$

$$=\left(\frac{n-1}{n}\right)x^2B_{n-2}(f_0)(x)+\frac{x}{n}=\left(1-\frac{1}{n}\right)x^2+\frac{x}{n}$$

$$B_n(f_2)\longrightarrow f_2$$

$\therefore B_n(f_i) \to f_i \quad \forall\, i = 0,1,2$

using Korovkin theorm -1

$\Rightarrow B_n(f) \to f$

uniformly on $[0,1]$

Remark: For any cont function $f:[a,b]\to\mathbb{R}$, by using $f\circ\phi$ where the fuction $\phi:[0,1]\to[a,b]$ is

$$\phi(x) = (b-a)x + a \quad \forall\, x\in[0,1]$$

so the above statement become for $C([a,b])$

The expansion of type $a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$ for $n\in\mathbb{N}$ are called trigonometric polynomials where $a_0, a_1, \ldots b_1, b_2, \ldots$ are real numbers. They are used to approximate real valued 2π-periodic integrable function. If function f is defined on $[-\pi,\pi]$ which is integrable and $f(-\pi) = f(\pi)$ then we define fourier coefficients of f as follows:

$$a_0(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt$$

$$b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt \qquad k\in\mathbb{N}$$

The series $a_0(f) + \sum(a_k(f)\cos kx + b_k(f)\sin kx)$ of fuctions defined on $[-\pi,\pi]$ is called fourier series of f.

eg: $f:[-\pi,\pi]\to\mathbb{R}$

$f(x) = \sin x$

$$a_0(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin t\,dt = 0$$

$$a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin t\cos kt\,dt = 0$$

$k \neq 0$

$$b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(1-k)t - \cos(k+1)t\,dt = 0$$

$$b_1(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin t\sin t\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}1-\cos 2t\,dt = 1$$

$\therefore$ $\sin x$ fourier is $\sin x$

Theorm 7.15: (Korovkin-2)

Let $X = \{f\in C[-\pi,\pi] : f(-\pi) = f(\pi)\}$ suppose $f_0(x) = 1$, $f_1(x) = \cos x$ & $f_2(x) = \sin x$ $\forall\, x\in[-\pi,\pi]$. Let $P_n: X\to X$ be linear map s.t $P_n(f)\geq 0$ if $f\geq 0$ — ⊕

if $P_n(f_i)\to f_i$ uniformly $\forall\, i=0,1,2$ then $P_n(f)\to f$ uniformly on $[-\pi,\pi]$ $\forall\, f\in X$

proof: If $f\in X$, $f = \operatorname{Re} f + i\operatorname{Im} f$ where $\operatorname{Re} f, \operatorname{Im} f \in X$ are real valued.

$P_n(f) = P_n(\operatorname{Re} f) + i P_n(\operatorname{Im} f)$ $\forall\, n\in\mathbb{N}$ is enough to prove that $P_n(f)\to f$ uniformly when f is real-valued. Let $f\in X$ be real valued. f is bounded, i.e $\exists\,\alpha\in\mathbb{R}$ s.t $|f(x)|\leq\alpha$ $\forall\, x\in[-\pi,\pi]$

$\therefore -2\alpha \leq f(x) - f(y) \leq 2\alpha$ —— (a)

let $\varepsilon > 0$, as f is uniformly cont on $[-\pi, \pi]$, $\exists \delta > 0$ s.t for $x, y \in [-\pi, \pi]$, $|x-y| < \delta$

$\Rightarrow -\varepsilon < f(y) - f(x) < \varepsilon$ —— (b)

fix $x \in [-\pi, \pi]$ and define $f_x(y) = \sin^2\left(\frac{y-x}{2}\right) \; \forall y \in [-\pi, \pi]$

for $|x-y| \geq \delta \Rightarrow f_x(y) \geq \sin^2 \delta/2$

combine (a), (b) to get

$$-\varepsilon - 2\alpha \leq f(y) - f(x) \leq \varepsilon + 2\alpha$$

$$\Rightarrow -\varepsilon - \frac{2\alpha f_x(y)}{\sin^2 \delta/2} \leq f(y) - f(x) \leq \varepsilon + \frac{2\alpha f_x(y)}{\sin^2 \delta/2}$$

using (#)

$$\Rightarrow -\varepsilon P_n(f_0) - \frac{2\alpha}{\sin^2\delta/2} P_n(f_x) \leq P_n(f) - f(x) P_n(f_0) \leq \varepsilon P_n(f_0) + \frac{2\alpha}{\sin^2 \delta/2} P_n(f_x)$$

$\Rightarrow P_n(f_0) \longrightarrow f_0$ uniformly, so $\exists\, m \in \mathbb{N}$ s.t

$$|P_n(f_0)(y) - f_0(y)| < \varepsilon/\alpha \quad \forall n \geq m \quad \forall y \in [-\pi, \pi]$$

$$|f(x) P_n(f_0)(x) - f(x)| \leq \alpha |P_n(f_0)(x) - f_0(x)| < \varepsilon$$

$$f_n = \frac{1}{2}\left(f_0 - \cos x f_1 - \sin x f_2\right)$$

$$P_n(f_x) = \frac{1}{2}\left(P_n(f) - \cos^2 x - \sin^2 x\right) = 0$$

$\exists\, n_0 \in \mathbb{N}$ s.t

$$\left|\frac{2\alpha}{\sin^2 \delta/2} P_n(f_x)(x)\right| < \varepsilon/2 \quad \& \quad \left|P_n(f_0)(x) - 1\right| < 1/2$$

$$1/2 \leq P_n(f_0)(x) \leq 3/2$$

$$\Rightarrow -\varepsilon/2 - \varepsilon/2 \leq P_n(f)(x) - f(x) P_n(f_0)(x) \leq 3\varepsilon/2 + \varepsilon/2$$

$n \geq \max\{n_0, m\} \Rightarrow P_n(f) \longrightarrow f$ uniformly on $[-\pi, \pi]$

$-3\varepsilon \leq P_n(f) - f \leq 3\varepsilon$

let us consider complex valued integrable function f on $[-\pi, \pi]$ s.t $f(-\pi) = f(\pi)$ then

with formula $2\cos nx = e^{inx} + e^{-inx}$

$2i \sin nx = e^{inx} - e^{-inx}$

fourier series f_∞ $(a_n, b_n \in \mathbb{C})$ is:

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = a_0 + \sum_{n=1}^{\infty} \alpha_n e^{inx} + \beta_n e^{-inx}$$

$$\left(a_n \left[\frac{e^{inx} + e^{-inx}}{2}\right] + b_n \left[\frac{e^{inx} - e^{-inx}}{2}\right] = \alpha_n e^{inx} + \beta_n e^{-inx} \right)$$

$$\alpha_n = \frac{a_n}{2} + \frac{b_n}{2i}, \quad \beta_n = \frac{a_n}{2} - \frac{b_n}{2i}$$

$$\alpha_n = (a_n - ib_n)/2 = \frac{a_n}{2} + \frac{b_n}{2i}$$

$$\beta_n = (a_n + ib_n)/2 = \frac{a_n}{2} - \frac{b_n}{2i}$$

if $\alpha_0 = a_0$ $\alpha_{-n} = \beta_n$

then we get

fourier series as $\sum_{n=-\infty}^{\infty} \alpha_n e^{inx}$

where now $\alpha_n = \frac{a_n}{2} + \frac{b_n}{2i} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\left[\frac{\cos nx}{2} - \frac{i\sin nx}{2}\right]dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\left[\frac{1}{2}\right]e^{-inx}\,dx$$

$$\hat{f}(n) = \alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$$

Note: for $n = 0, 1, \ldots$ the n^{th} partial sum is

$$S_n(x) := \sum_{k=-n}^{n} \hat{f}(k)e^{ikx}$$

Note: There exist a continuous function f on $[-\pi,\pi]$ with $f(-\pi)=f(\pi)$ s.t $\{S_n(x)\}_{n=0}^{\infty}$ diverges at some point in $(-\pi,\pi)$

Some formulas:

fourier of $f(x)$:

$$a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1} b_k \sin kx = \sum_{n=-\infty} \alpha_n e^{inx}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt \qquad \alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt$$

7th Nov :

Defn: n^{th} Dirichlet kernel: for $n=0,1,\ldots$ define n^{th} dirichlet kernel by $D_n(x)=\sum_{-n}^{n} e^{ikx}$ for $x\in\mathbb{R}$. Then for $x\in[-\pi,\pi]$ we have

$$\left(\begin{aligned} f(x) &= \sum_{k=-\infty}^{\infty} \alpha_n e^{ikx} \\ \alpha_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt \end{aligned}\right)$$

$$S_n(x) = \frac{1}{2\pi}\sum_{k=-n}^{n}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt\; e^{ikx}$$

$$\left(\frac{1}{2\pi}\sum_{-n}^{n}\alpha_n e^{inx}\right)$$

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,D_n(x-t)\,dt$$

where $D_n(x-t)=\sum_{k=-n}^{n} e^{ikx-ikt}$

$$D(x)=\sum_{-n}^{n} e^{ikx}$$

$$D_n(x-t)=\sum_{-n}^{n} e^{ikx-ikt}$$

we consider the arithemetic means $\sigma_n(f)(x)=\left(\sum_{k=0}^{n-1} S_k(x)\right)/n \quad \forall n\in\mathbb{N}$

$$S_n(f) \xrightarrow{\text{Pt. wise}} f$$

$$\text{but } \sigma_n(f) \xrightarrow[\text{unif}]{} f$$

$$\sigma_n(f)(x)=\left(\sum_{k=0}^{n-1} S_k(x)\right)/n \quad \forall n\in\mathbb{N}$$

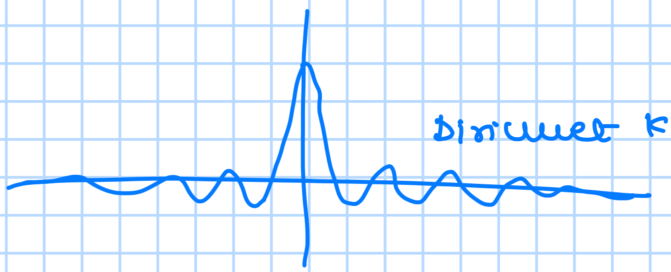
Defn: Fejer kernel:

$$K_m(x) = \frac{1}{m}\sum_{k=0}^{m-1} D_k(x) \quad x\in\mathbb{R}$$

$x\in[-\pi,\pi]$ we have

$$\sigma_m(f)(x) = \frac{1}{m}\sum_{k=0}^{m-1}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_k(x-t)\,dt$$

$$\sigma_m(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,K_m(x-t)\,dt$$



Dirichlet kernel for large n.

Remark: let $\sum_{n=0}^{\infty} c_n$ be a series of numbers. Let $S_n=\sum_{n=0}^{n} c_n$ be partial sum

The n^{th} cesaro sum of the series $\sum c_n$ is defined as the arithmetic mean.

$$\sigma_n = \frac{S_0+S_1+\cdots S_{n-1}}{n}$$

consider the series $\sum_{n=0}^{\infty}(-1)^n$, so the sequence of partial sum is $(1,0,1,\ldots)$ which does not converge. But the seq $\sigma_n=\left(\frac{1}{1},\frac{1}{2},\frac{2}{3},\frac{2}{4},\frac{3}{5},\ldots\frac{4}{8}\ldots\right)$ conv to $\frac{1}{2}$.

Lemma 7.16: $K_n(x) = \frac{1}{n}\frac{\sin^2(nx/2)}{\sin^2(x/2)}$

3

proof: $D_K(x) = \sum_{n=-k}^{k} e^{inx} = \underset{1}{e^{-ikx}} + \underset{2}{e^{-ikx+ix}} + e^{-ikx+2ix} + \dots \dots + e^{0} + e^{ix} + \dots e^{ikx}$

$(e^{-ikx+2ikx})$

$\underline{2k+1}$

$$= e^{-ikx}\left[\frac{r^{n}-1}{r-1}\right] \qquad r = e^{ix}$$

$$= e^{-ikx}\left[\frac{e^{i2kx+ix}-1}{e^{ix}-1}\right]$$

$$= \frac{e^{ikx+ix} - e^{-ikx}}{e^{ix}-1}$$

$$K_m(x) = \frac{1}{m}\sum_{m=0}^{m-1} D_m(x)$$

$$= \frac{1}{m}\sum_{m=0}^{m-1}\left[\frac{e^{imx+ix} - e^{-imx}}{e^{ix}-1}\right]$$

$$= \frac{1}{m}\frac{1}{e^{ix}-1}\left[\frac{e^{ix}[e^{imx}-1]}{e^{ix}-1} - \frac{(1-e^{-imx})}{(1-e^{-ix})}\right]$$

$$= \frac{1}{m}\frac{1}{(e^{ix}-1)^2}\left[e^{i(m+1)x} - e^{ix} - e^{ix} + e^{i(m+1)x}\right]$$

$$= \frac{1}{m}\frac{\sin^2 mx/2}{\sin^2 x/2}$$

Theorm 7.16: (Fejer Theorm) Let f be a cont $\mathbb{C}$- valued function on $[-\pi,\pi]$ s.t $f(\pi) = f(-\pi)$. Then the sequence of arithmetic means of the partial sum of the fourier series of f converges to f on $[-\pi,\pi]$ uniformly.

proof: Let $X = \{f \in C([-\pi,\pi]) \mid f(\pi) = f(-\pi)\}$. For $f \in X$ and $m \in \mathbb{N}$, the map σ_m is a linear map. Also, since $K_m(x-t) \geq 0 \ \forall x, t \in [-\pi,\pi]$

$\sigma_m(f)(x) \geq 0 \ \forall x \in [-\pi,\pi]$
when $f(t) \geq 0$

$$\sigma_n(f)(x) = \left(\sum_{k=0}^{n-1} S_k(x)\right)/n$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\underbrace{K_m(x-t)}_{\geq 0}\,dt \quad \text{as } K_m(x) = \frac{1}{m}\frac{\sin^2 mx/2}{\sin^2 x/2}$$

$f_0(t)=1 \quad f_1(t)=\cos t \quad f_2(t)=\sin t \quad t\in[\pi,\pi]$

$1,\ \cos t = \dfrac{e^{it}+e^{-it}}{2},\ \sin t = \dfrac{e^{it}-e^{-it}}{2i}$

$$\sigma_m(f_0)(t) = \frac{1+1+1+\cdots 1}{m} = 1$$

$$\sigma_m(f_1)(t) = \frac{S_0+S_1+\cdots S_{m-1}}{m}$$

$$= \frac{0+\cos t+\cdots+\cos t}{m}$$

$$= \left(\frac{m-1}{m}\right)\cos t$$

$$\sigma_m(f_2)(t) = \frac{0+\sin t+\cdots \sin t}{m} = \left(\frac{m-1}{m}\right)\sin t$$

for $\forall m\in\mathbb{N}$, $t\in[-\pi,\pi]$ the seq. $\{\sigma_m(f_i)\}_{m=1}^{\infty}$ conv to f_i for $i=0,1,2$. $\therefore$ By Korovkin's second theorem, $\forall f\in X$ the seq $\{\sigma_m(f)\}_{m=1}^{\infty}$ conv to f uniformly on $[-\pi,\pi]$

<u>Revision question</u> : find the radius of convergence of $\sum_{n=0}^{\infty} 3^n x^{2n}$

$\lim_{n\to\infty} \sup |3^n x^{2n}|^{1/n} = 3|x|^2$

By root test

$\Sigma\, 3^n x^{2n}$ conv iff $3|x^2|<1$

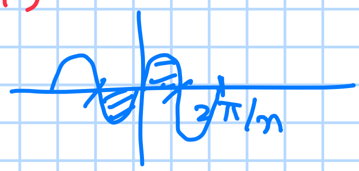
$\Rightarrow |x| < \frac{1}{\sqrt{3}}$

div if $3|x|^2>1 \Rightarrow |x|>\frac{1}{\sqrt{3}}$

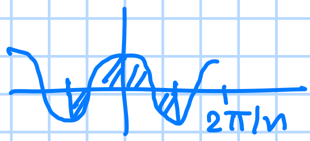
so, radius of convergence of $\Sigma 3^n x^{2n}$ is $r=\frac{1}{\sqrt{3}}$

<u>Note</u>: (Some useful integrals for fourier transformation)

$\int_{-\pi}^{\pi} \sin nx\,dx = 0 \qquad \int_{-\pi}^{\pi} \cos nx\,dx = 0$ for $n\neq 0$



$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 0 & ; n\neq m \\ \pi & ; n=m \end{cases}$$



$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = \begin{cases} 0 & ; n\neq m \\ \pi & ; n=m \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx = 0$$

$$\int_{-\pi}^{\pi} e^{inx}\,dx = \begin{cases} 2\pi & ; \quad n=0 \\ 0 & ; \quad n\neq 0 \end{cases}$$