MA403
114

Ref : Pricipalas of matematical andysis - Walter Rudin
Mathematical analysis - T. Apostal

Coms : Moodle

Eval :

$$\left.\begin{array}{l}\text{Quizes} - 15 \times 2\\ \text{midsem} - 25 \times 1\\ \text{Endsem} - 45 \times 1\end{array}\right\}\text{Theory + Problems}$$

29th July 2024 :

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, \ldots\}$ set of integers

define : $\mathbb{Q}^+ = \{ p/q : p,q \in \mathbb{Z}, (p.q)>0, \gcd(p,q)=1 \}$

$\mathbb{Q} = -\mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^+$

$\mathbb{Q}$ is a field, $\mathbb{R}$ is also a field

ordered field → see → see down

$+ -$ group/comm groups
$+ - \times$ Ring/comm rings
$+ - \times \div$ Fields $\mathbb{Q}$, $\mathbb{Z}/p\mathbb{Z}$
inf. finite
prime field
F will have only one

$\mathbb{R}$ - set of real numbers
$\mathbb{Q}$ - set of rational numbers

(Different defination using equivalance relations:
$\frac{p_1}{q_1} \sim \frac{p_2}{q_2}$ $p_1 q_2 = q_1 p_2$
Equivalance class sets → rational numbes.)

$S = \{(x,y) \in \mathbb{Z} \times \mathbb{Z}, y \neq 0\}$

rational no. to be eq class of elements of S under eq relation
$(a,b) \simeq (c,d) \Leftrightarrow ad = bc$
① $(a,b) \simeq (a,b)$ yes
② $(a,b) \simeq (c,d)$ then $(c,d) \simeq (a,b)$ yes
③ $(a,b) \simeq (c,d)$ and $(c,d) \simeq (e,f)$ $\Rightarrow (a,b) \simeq (e,f)$

Addition $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
$(x,y) \mapsto x+y$

Multiplication $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
$(x,y) \mapsto x \cdot y$

} Binary operations

group : closed, assoc, inverse, identity
+ comm → comm group

Rings : $\boxed{\begin{array}{l} + \text{ comm} \\ \times \text{ not comm okay} \end{array}}$ closed ⊗ assoc ⊗ dis ⊗

see identity → Ring with id.

Field axioms :

→ $(\mathbb{R}, +, \cdot)$ satisfies the following :

field { (i) $(\mathbb{R}, +)$ is a commutitive group with 0 as additive identity
(ii) $(\mathbb{R}\setminus\{0\}, \cdot)$ is a commutive group with 1 as multiplicative identity.
(iii) (Distributive) $x(y+z) = x \cdot y + x \cdot z$ $\forall x, y, z \in \mathbb{R}$

$(\mathbb{R}, +, \cdot)$ is a field → Any set satisfyg this property.

$(\mathbb{R}, +)$ closed under addition, has add inv, assosiativity, identity
$\left.\begin{array}{l} x \in \mathbb{R} \\ y \in \mathbb{R} \end{array}\right\} \Rightarrow x+y \in \mathbb{R}$ $x + (-x) = 0$ $x+(y+z) = (x+y)+z$ $x+0 = x = 0+x$

$(\mathbb{R}\setminus\{0\}, \cdot)$ closed under multi, has mul inv, assosiativity, identity
$\left.\begin{array}{l} x \in \mathbb{R} \\ y \in \mathbb{R} \end{array}\right\} \Rightarrow x \cdot y \in \mathbb{R}$ $x \cdot \frac{1}{x} = 1$, $\frac{1}{x} \to x^{-1}$ $x(y \cdot z) = (xy)z$ $x \cdot 1 = x = 1 \cdot x$

] group

Commutative
$x+y = y+x$
$xy = yx$

] commutative group

Order axioms: (order relⁿ $<$) → see this / ordered set / ordered field.

satisfies following iv is ordered field

(i) if $x, y \in \mathbb{R}$, then $x<y$, $x=y$, $y<x$
($x>y$ means $y<x$)

(ii) if $x<y$, $y<z$, then $x<z$

(iii) if $x<y$, then $x+z<y+z$, for $x, y \in \mathbb{R}$

(iv) if $x, y \in \mathbb{R}$, $x>0$, $y>0$, then $x \cdot y > 0$
(If $x>y$ and $z>0$ then $xz > yz$)

Examples: $\mathbb{Q}$, $\mathbb{R}$.

Completeness axioms → see

let S be a non-empty subset in $\mathbb{R}$.
s.t $\exists\, b \in \mathbb{R}$, $\forall x \in S$, $x \leq b$. (S is bounded above)
then there exist $\sup S$.

Note:

	field	$\mathbb{R}$	$\mathbb{Q}$
Field axioms		✓	✓
order axioms		✓	✓
completeness axioms		✓	✗

$\mathbb{Q}$ does not satisfy completeness axiom
eg: $E = \{x \in \mathbb{Q};\ x>0,\ x^2<2\}$

then let $\beta = 2$
as $\beta^2 = 4$
$\beta^2 \geq x^2$
so upper bound $=2$
but not $\sup E$
$s = \sup E \in \mathbb{Q}$
as $s^2 \neq 2$ $\quad r = \frac{2s+2}{s+2}$
(proof next) left

$\mathbb{R} - \mathbb{Q}$ → irrational numbers.

If $x \in \mathbb{R}$ and $x>0$, then x is said to be positive
set of all positive real numbers will be defined by $\mathbb{R}^+$

If $x \in \mathbb{R}$ and $x<0$, then x is said to be negative
set of all neg. real numbers will be defined by $\mathbb{R}^-$

$(a,b) := \{x \in \mathbb{R};\ a<x<b\}$ open interval
$[a,b] := \{x \in \mathbb{R};\ a \leq x \leq b\}$ closed interval

$(-1,2)$
$(3,7)$
$[3,2]$

Also: $(a,+\infty) := \{x \in \mathbb{R};\ x>a\}$, $(-\infty, a) := \{x \in \mathbb{R};\ x<a\}$

$[a,\infty)$, $(-\infty,a]$, $[a,b)$, $(a,b]$ → see definitions (next) left

Note: $-\infty, \infty$ are symbols; not real numbers.

Note: B/w two rationals there are inf. many rational numbers.

proof sketch: $a, b \to \frac{a+b}{2}$ b/w a, b and keep repeating this. (given $a \neq b$)

$$\begin{aligned} a &< b \\ \Rightarrow a+a &< a+b \\ \Rightarrow a &< \frac{a+b}{2} \end{aligned} \qquad \begin{aligned} a+b &< b+b \\ \Rightarrow \frac{a+b}{2} &< b \end{aligned}$$

$$\Rightarrow a < \frac{a+b}{2} < b$$

So we cannot ask for "next larger" rational number.

Defn: Let S be a non-empty set of real numbers

$\exists\, b \in \mathbb{R}$ s.t $x \leq b \;\forall\; x \in S$, then b is called an upper bound for S. In this case S is bounded above by b.

If the upperbound $b \in S$ then b is called the maximum element of S. ($\max S = b$)

If S does not have a upper bound, then S is not bounded above.

Def: Similarly we define lower bound, bounded below, and min element.

Remarks: (i) Maxima and minima elements are unique.
(ii) Some sets like $[1,2)$ are bounded above but not a max element.

For such sets: this is useful:

Defn: Let S be a non-empty subset of real numbers which is bounded above. A real number b is called least upper bound (lub) or supremum if it has the following properties:

(a) b is an upper bond of S.

(b) No number less than b is an upper bond for S.

Proof of $\mathbb{Q}$ is not commt.

Proof:
let $E = \{x \in \mathbb{Q} : x^2 < 2\}$
then E is a non-empty subset of $\mathbb{Q}$
as $1 \in E$
and $1 \in \mathbb{Q}$

now, it also has an upper bound 2.
as $x^2 < 2$
$x^2 < 2^2 = 4$
as $2 < 4$
$x^2 < 4$
so $\forall x \in E$, $x < 2$

now, let $x' = \sup E$, then

DD

30th July 2024:

- field and order axioms $(\mathbb{R},+,\cdot)$, $(\mathbb{Q},+,\cdot)$
- least upper bound (lub) or supremum (sup)

$(S \subseteq \mathbb{R}, S \neq \phi, b = \sup S)$ eg: $[1,2)$ does not have max element but $\sup[1,2) = 2$

↗ Similarly we can define greatest lower bound (glb) or (inf) infimum. (glb is opp. lub)

Note: sup and inf are unique for $S \subseteq \mathbb{R}$.

Done see

Proof of $\mathbb{Q}$ is not comp:

see proof

Completeness axiom:

Every non-empty subset S of $\mathbb{R}$, which is bounded above has a sup.

Remarks: Observe that the set $S_1 = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}$ does not have a supremum (in $\mathbb{Q}$)

idea: $\alpha^2 < 2$ ✱, $\alpha^2 > 2$ ✱, $\alpha^2 = 2$ (not in $\mathbb{Q}$)

Theorm 1.1: Let S be a non-empty set of real numbers which has a supremum, say $s = \sup S$. Then $\forall a \in \mathbb{R}$, s.t $a < s$, $\exists$ some $x \in S$ s.t $a < x \leq s = \sup S$

proof: let $a < s$.

If $a \geq x \ \forall x \in S$,

then a will be an upper bounded for S.
But $a < s$, which is a contradiction.

so, $a \geq x \ \forall x \in S$ ✱

$\therefore \exists x \in S$ s.t $a < x \leq s$

Theorm 1.2: The set $\mathbb{Z}^+ (= \mathbb{N})$ is not bounded above.

proof: If $\mathbb{Z}^+$ is bounded above then it has a supremum, say S.

By theorm 1.1, $\exists n \in \mathbb{Z}^+$ s.t $s-1 < n \leq s$ $(a = s-1)$

$s-1 < n \leq s$
then $s < n+1$,
but as $n+1 \in \mathbb{Z}^+$
so, s is not the supremum,
this is a contradiction.
$\mathbb{Z}^+$ is bounded above ✱
$\mathbb{Z}^+$ is not bounded above

Theorm 1.3: (Archimedean property)
If $x > 0$ and $y \in \mathbb{R}$, then $\exists n_0 \in \mathbb{N}$ s.t. $n_0 x > y$.

proof: let $x > 0$ and $y \in \mathbb{R}$.
now if $\frac{y}{x} \geq n \ \forall n \in \mathbb{N}$, then $\frac{y}{x}$ is upper bound for $\mathbb{N}$

which is a contradiction (theorm 1.2)

so $\frac{y}{x} \geq n \ \forall n \in \mathbb{N}$ ✱

$\exists n_0 \in \mathbb{N}$, $n_0 > \frac{y}{x} \Rightarrow \exists n_0 \in \mathbb{N}$ s.t $x n_0 > y$.

(not syllabus) — x — x — x — x — x — x —

Defination of real numbers : → see using dedoteind cut / other method.

Defn: A Dedekind cut $A|B$ in $\mathbb{Q}$ is a pair of subsets A and B of $\mathbb{Q}$ st.

(i) $A \neq \phi$, $B \neq \phi$, $A \cap B = \phi$, $A \cup B = \mathbb{Q}$

(ii) If $a \in A$ and $b \in B$, then $a < b$
(A contains no largest elements)

Defn: A real number is a dedekind cut in $\mathbb{Q}$.

e.g. (i) $A = \{x \in \mathbb{Q} \mid x < 2\}$, $B = \{x \in \mathbb{Q} \mid x \geq 2\}$

(ii) $A = \{x \in \mathbb{Q} \mid x < 0 \text{ or } x^2 < 2\}$, $B = \{x \in \mathbb{Q} \mid x \geq 0 \text{ and } x^2 \geq 2\}$

$A|B$ and $C|D$, then   see this from Rudin (Appendix Chap 1)

$$A|B + C|D = E|F \qquad E = \{a + c \mid a \in A \text{ and } c \in C\}$$

$$F = \mathbb{Q} - E = E^c$$

Cauchy approach : let F be ordered field. let e be a set of all cauchy sequences in F and N be set of all null sequence
see null seq.
see ←
Then $E = F/N$

see from: Real and Abstract Analysis
Hewitt and stromberg (section 1.5)

Proof of $\mathbb{Q}$ not complete :

let $E = \{q \in \mathbb{Q} \mid q^2 < 2\} \subset \mathbb{Q}$

now, as $0 \in E$, E is non-empty

also, if $q \geq 2 \Rightarrow q^2 \geq 4 > 2$
so 2 is upper bound.

$\therefore E$ is non-empty and has upper bound → should have sup.
let $\sup = \alpha$

Note :
$1 \leq \alpha \leq 2$ cause $1 \in E$
$\quad\quad$ ↳ upper bound

for $\alpha^2 = 2$ (prove $\alpha^2 < 2$ not possible and $\alpha^2 > 2$ not possible)

if $\alpha^2 < 2$: then α cannot be an upper bound for E

$$q^2 = (\alpha + \varepsilon)^2 = \alpha^2 + 2\alpha\varepsilon + \varepsilon^2$$

also as $\alpha \leq 2$

$$q^2 \leq \alpha^2 + 4\varepsilon + \varepsilon = \alpha^2 + 5\varepsilon$$

this is $\alpha^2 + 5\varepsilon < 2$ if

$\underline{\varepsilon < 2 - \alpha^2}$

so $q = \alpha + \frac{2-\alpha^2}{5}$ is a rational in E strictly larger than α

so α is not an upper bound for E.

if $\underline{\alpha^2 > 2}$: $q^2 = (\alpha - \varepsilon)^2 = \alpha^2 - 2\alpha\varepsilon + \varepsilon^2$

as $\alpha \leq 2$ $\quad -\alpha \geq -2$

$q^2 = \alpha^2 - 2\alpha\varepsilon + \varepsilon^2 \geq \alpha^2 - 4\alpha\varepsilon + \varepsilon^2 > \alpha^2 - 4\varepsilon$

if $\alpha^2 - 4\varepsilon > 2 \Rightarrow \varepsilon < \frac{\alpha^2 - 2}{4}$

for $q = \alpha - \left(\frac{\alpha^2-2}{6}\right)$ rational smaller than α

now as $\tilde{q} < \alpha$ but still $\tilde{q}^2 > 2$,

$\tilde{q}$ is also an upper bound.

$\therefore$ α is not lub

as $\alpha^2 < 2$ and $\alpha^2 > 2$ are contradictions,

$\alpha^2 = 2.$

now let $\alpha = \frac{m}{n}$, $\left(\frac{m}{n}\right)^2 = 2$

$m^2 = 2n^2$

2 is a factor of m as it is a factor of m^2

$2 \mid m$, so $m = 2k$

$4k^2 = 2n^2$

$2k^2 = n^2$

so, $2 \mid n$, but as $\gcd(m,n) = 1$

$*$.

$\therefore \alpha^2 = 2 \Rightarrow \alpha \notin \mathbb{Q}$

$\therefore$ $\mathbb{Q}$ is not complete.

1st Aug :

- Field, order axioms, supremum
- $\mathbb{N}$ is not bounded. → Proof by contradiction

theorm 1.1 → theorm 1.2

suppose bounded

$n \le x \ \forall x \in \mathbb{N}$

$s-1 < n \le s$

$s < n+1$

as $n+1 \in \mathbb{N}$

- Archemedian theorm 1.3
- Dedekind cut $A|B$ → Real number $A, B \subseteq \mathbb{Q}$

$A|B < C|D$ if $A \subsetneq C$

Theorm 1.4 : (Cauchy-Schwarz inequality)

if $a_1, a_2, \dots a_n$ and $b_1, b_2, \dots b_n$ are arbitrary real numbers, then

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} b_k^2\right)\left(\sum_{k=1}^{n} a_k^2\right)$$

Moreover if some $a_i \neq 0$, then equality holds iff $\exists$ a real number x s.t $a_k x + b_k = 0 \ \forall k = 1, 2, \dots n$ (Note: the same x)

$(a_1, a_2 \dots a_n) = c(b_1, b_2, \dots b_n)$ (lin ind)

or all $a_i = 0$

Equality holds :

① all $a_i = 0$

② if $a_i \neq 0$, then equality holds iff $\exists x \in \mathbb{R}$ s.t $a_k x + b_k = 0 \ \forall k = 1, 2, \dots n$

Proof: $\sum_{k=1}^{n} (a_k x + b_k)^2 \ge 0$

and equality holds if each term $a_k x + b_k = 0$

so, $\underbrace{\left(\sum a_k^2\right)}_{A} x^2 + 2\underbrace{\left(\sum a_k b_k\right)}_{C} x + \underbrace{\left(\sum b_k^2\right)}_{B} \ge 0$

$Ax^2 + 2Cx + B \ge 0$

if $A = 0$, then equality holds.

as $\underset{0}{(\sum a_k b_k)^2} \underset{=}{\le} \underset{0}{(\sum a_k^2)(\sum b_k^2)}$

now if $A \neq 0$, Then with $x = -\frac{C}{A}$ we get:

$A\left(\frac{-C}{A}\right)^2 + 2C\left(\frac{-C}{A}\right) + B \ge 0$

$$\frac{C^2}{A} - \frac{2C^2}{A} + B \geq 0$$

$$-\frac{C^2}{A} + B \geq 0$$

$$B \geq \frac{C^2}{A}$$

$$C^2 \leq AB$$ (Cauchy swarts inequality)

so, $\left(\Sigma a_k^2\right)\left(\Sigma b_k^2\right) \geq \left(\Sigma a_k b_k\right)^2$ → Here we reverse our steps to get to $\sum_{k=1}^{n}(a_k x + b_k)^2 \geq 0$

then put $=0$ to get $\exists\, x \in \mathbb{R}$ s.t. $a_k x + b_k = 0$ $\forall a_k, b_k$

If $a_i \neq 0$ for some i, equality holds in C.S. inequality iff $Ax^2 + 2Cx + B = 0$ for $x = -\frac{C}{A}$

iff $a_k x + b_k = 0 \;\forall\, k = 1, 2, \ldots n$

At this point do this week's homework.

Metric spaces

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \mathbb{R}$ — n copies

$= \{\underbrace{(x_1, x_2, \ldots x_n)}_{\text{sequence}} \mid x_i \in \mathbb{R}\}$

} n-dimensional euclidean space

Here, we have for $x \in (x_1, \cdots x_n)$ and $y = (y_1, \ldots y_n)$

(i) $x = y$ iff $x_k = y_k \;\forall\, k = 1, \ldots n$
(ii) $x + y = (x_1 + y_1, \ldots x_n + y_n)$
(iii) $ax = (ax_1, ax_2, \ldots ax_n) \;\forall\, a \in \mathbb{R}$
(iv) $-y = (-y_1, -y_2, \ldots -y_n)$
(v) $0 = (0, 0, \ldots 0)$

Def^n: (i) The inner product or dot product b/w $x, y \in \mathbb{R}^n$ is defined as

$$x \cdot y = \sum_{k=1}^{n} x_k y_k$$

(ii) The norm of $x \in \mathbb{R}^n$ is defined as

$$\|x\| = \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}}$$

The Cauchy-Swarz ineq. can be written as

$$|x \cdot y| \leq \|x\| \, \|y\|$$

as $\left(\Sigma x_i y_i\right)^2 \leq (\Sigma x_i)^2 (\Sigma y_i)^2$

$$\sqrt{(\Sigma x_i y_i)^2} \leq \sqrt{(\Sigma x_i)^2 (\Sigma y_i)^2}$$

$$\underbrace{|x \cdot y|}_{\text{mod not norm as real number}} \leq \|x\| \, \|y\|$$ → vector so norm

Theorm 2.1(i): Let $x, y \in \mathbb{R}^n$, Then

(i) $\|ax\| = |a|\,\|x\|$
(ii) $\|x+y\| \leq \|x\| + \|y\|$

proof: (i) $\|ax\| = \|a(x_1, x_2 \dots x_n)\| = \|(ax_1, ax_2 \dots ax_n)\|$

$$= \left(\sum_{k=1}^{n} (ax_k)^2\right)^{\frac{1}{2}} = |a| \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}} = |a|\,\|x\|$$

(ii) $$\|x+y\|^2 = \sum_{k=1}^{n} (x_k + y_k)^2 = \sum_{k=1}^{n} x_k^2 + 2\sum_{k=1}^{n} x_k y_k + \sum_{k=1}^{n} y_k^2$$

$$= \|x\|^2 + 2x\cdot y + \|y\|^2$$

$$\leq \|x\|^2 + 2|x\cdot y| + \|y\|^2$$

cauchy swartz ineq.

$$\leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

so $\|x+y\|^2 \leq (\|x\| + \|y\|)^2$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$

Def^n: A metric space is a non-empty set M together with the function $d: M \times M \to \mathbb{R}$ satisfying the following:

(i) $d(x,x) = 0 \quad \forall x \in M$
(ii) $d(x,y) \geq 0 \quad \forall x, y \in M$
(iii) $d(x,y) = d(y,x) \quad \forall x, y \in M$ (Symmetric)
(iv) $d(x,y) + d(y,z) \geq d(x,z) \quad \forall x, y, z \in M$

The map d is called a metric.

Theorm 2.2:
$\mathbb{R}^n$ with $d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as $d(x,y) = \|x-y\|$ is a metric space.

proof: $d(x,x) = \|x - x\| = 0$
$d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$
$d(x,y) = (\Sigma(x_k - y_k)^2)^{1/2} \geq 0$

Theorm 2.1 (ii): **Proof**:
in 2.1(i) replace x by $x-y$ and y by $y-z$ to get
$\|(x-y) + (y-z)\| \leq \|x-y\| + \|y-z\|$
$\Rightarrow \|x-y\| + \|y-z\| \geq \|x-z\|$
$\Rightarrow d(x,z) \leq d(x,y) + d(y,z)$

5th Aug : Recap :

- $\mathbb{R} \to$ ordered field with completness axiom.
- metric spaces $- (\mathbb{R}^n, d)$
- $(\mathbb{R}^n, d)$, $d(x,y) = \|x-y\|$ $\forall\, x,y \in \mathbb{R}^n$

$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

1. $d(x,y) \geq 0$
2. $d(x,x) = 0$
3. $d(x,y) = d(y,x)$
4. $d(x,y) + d(y,z) \geq d(x,z)$

Now: $(\mathbb{R}^n, d)$ is called the euclidean metric space. $(\|x-y\|)$

Examples of metric :

① discrete metric space : let M be a non-empty set and $d : M \times M \to \mathbb{R}$ be defined by

$$d(x,y) = \begin{cases} 1 ; & x \neq y \\ 0 ; & x = y \end{cases}$$

1. $d(x,y) \geq 0$
2. $d(x,x) = 0$
3. $d(x,y) = d(y,x)$ (symmetric)
4. $d(x,y) + d(y,z)$

Triangle inequality

$$\begin{cases} \text{if } x = y = z & \geq d(x,z) \\ x = y \neq z & \geq d(x,z) \\ x \neq y = z & \geq d(x,z) \\ x \neq y \neq z & \geq d(x,z) \end{cases}$$

$\therefore$ metric

This is called the discrete metric space.

Ex: let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $d(x,y) = |x_1 - y_1| + |x_2 - y_2| + \cdots \cdots |x_n - y_n|$

show that $(\mathbb{R}^n, d)$ is a metric space. $\to$ done

① $d(x,y) \geq 0$ $= d(y,x)$, $d(x,x) = 0$

② from $\|x+y\| \leq \|x\| + \|y\|$ $d(x,y) + d(y,z) \geq d(x,z)$

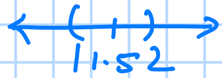
Def^n : let (X,d) be a metric space

(a) let $a \in X$ and $r > 0$ then the set $\{x : d(a,x) < r\}$ is called the open ball centered at a, and of radius r.

It is denoted by $B(a,r)$

example :

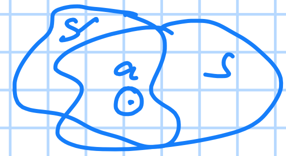
$B(1.5, 0.5) = (1,2)$ in $\mathbb{R}$ for $d(x,y) = \|x-y\|$



Eg : $B(a,r) = \{x \in \mathbb{R}^n \mid \|a - x\| < r\}$ is the open ball in $(\mathbb{R}^n, d)$

(b) let S be a subset of X and $a \in S$. Then a is called an interior point of S if $\exists\, r > 0$ s.t $B(a,r) \subseteq S$

* The set S is called a neighbourhood of a.



* The set of all interior points of S is called interior of S.

(c) A set $S \subseteq X$ is called an open set if all the points of S are interior points of S.

Eg: (1) $(1,2) \cup (4,5)$ is open set in $\mathbb{R}$

$\uparrow$ open ball $\uparrow$ open ball

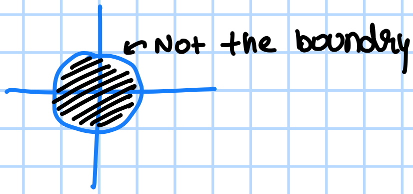
open ball$_1$ $\cup$ open ball$_2$ = open set

the point 1.75 is an interior point because $B(1.75, r) \subseteq S$

$\quad \parallel$ for $r = 0.25$

$(1.5, 2) \subseteq S$

(2) $B(0,1)$ is inside the unit circle in $\mathbb{R}^2$



$\{A_n \mid n \in \mathbb{N}\} = \{A_1, A_2, \ldots\} \leftarrow$ Example

Thm: The union of any collection of open sets is an open set in a metric space (X, d)

proof: let $F = \{A_\lambda \mid \lambda \in I\}$ be a collection of open sets in X.

let $B = \bigcup_{\lambda \in I} A_\lambda$ to show: B is open.

and $x \in B$

then $\exists \lambda_0 \in I$ s.t

$x \in A_{\lambda_0}$

since A_{λ_0} is an open set, $\exists r > 0$ s.t

$B(x, r) \subset A_{\lambda_0}$

so, $B(x, r) \subset B$ $\quad (\because A_{\lambda_0} \subset B)$

i.e x is an interior point of B. Hence B is an open set.

$\{1\}$ not open

$\bigcap \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ not open

$\therefore$ $\cap$ of open sets not nes. open.

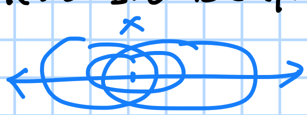
Thm: The intersection of any finite collection of open set is open.

proof: let $F = \{A_k \mid 1 \leq k \leq n\}$ be a finite collection of open sets.

suppose $C = \bigcap_{k=1}^{n} A_k$ and $x \in C$.

so, $x \in A_k, \forall k = 1, \ldots, n$.

Thus, $\exists r_k > 0$ s.t $B(x, r_k) \subset A_k, \forall k = 1, 2, \ldots n$



Define $r := \min\{r_1, r_2, \dots r_n\}$ (clearly $r>0$)

and $B(x,r) \subset \bigcap_{k=1}^{n} B(x, r_k)$ $\quad$ ($\because$ if $y \in B(x,r)$, $d(x,y) < r \leq r_k$ $\forall k = 1,2,\dots n$)

$\subset \bigcap_{k=1}^{n} A_k = C$

Thus x is an interior point of C.

Hence C is an open set.

6th Aug : Recap :

Open set (open ball, nbd, interior of a set)

nbd ↑ neighbourhood

Def^n: A subset S of X where (X,d) is a metric space is said to be closed if $X \setminus S = S^c$ is open.

Corollary 2.5 : The union of any finite collections of closed sets is closed, and the intersection of an arbitrary collection of closed set is closed.

proof : Assume $D = \bigcup_{k=1}^{n} A_k$ where A_k's are closed set

$D^c = \bigcap_{k=1}^{n} A_k^c$. since A_k^c are open sets, the set D^c is open. (By thm 2.4)

Similarly, one can prove the other statement.

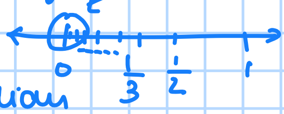
Def^n : (a) Let (X,d) be a metric space. Let S be a subset of X and let $p \in X$. The point p is said to be a limit point of S if every ball $B(p,r)$ has a point $q \neq p$ s.t $q \in S$.

Note : p may / may not be a point of S.

(b) If $p \in S$ is not a limit point of S, then p is said to be an isolated point.

Example : (a) Let $S = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$

limit point $= 0$ as



how small r we find some $\frac{1}{n}$ inside it.

for $n_0 > \frac{1}{r}$ $\quad$ $r > \frac{1}{n_0}$ archimedian property

for any $r > 0$

$\exists n$ s.t $n_0 r > 1$

$\Rightarrow r > \frac{1}{n_0}$

So $b(0,r) \ni \frac{1}{n_0}$

$\therefore$ 0 is a limit point

The points $\frac{1}{n}$ are isolated points as $\frac{1}{n} \in S$ but not limit points as $r < \left(\frac{1}{n} - \frac{1}{n+1}\right)$

(b) Consider $\mathbb{R}$ with discrete metric d and let $S = \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\}$

$$d(x,y) = \begin{cases} 1 ; & x \neq y \\ 0 ; & x = y \end{cases} \qquad B(0,r) = \left\{ x \,\middle|\, d(x,0) < r \right\}$$

The point $\frac{1}{12}$ not a limit point as for $r < 1$, $B\left(\frac{1}{12}, 0.5\right) = \left\{\frac{1}{12}\right\}$

Note: $\frac{1}{12}$ is an interior point as $\exists r$ s.t $B\left(\frac{1}{12}, r\right) \subseteq S$

(c) every real number is a limit point of the set $\mathbb{Q}$. → proof see

(d) In the set $B = [0,1] \cup \{2\}$, the point 2 is an isolated point

Theorm 2.6 : If x is a limit point of a set S, then every open ball of x contains infinitely many points of S.

proof : Suppose there is a ball $B(x,s)$ which contains (exactly) finitely many points of S. Let $y_1, y_2 \ldots y_n$ be those points of $S \cap B(x,s)$ which are distinct from x.

Let $r = \min\limits_{1 \le m \le n} d(x, y_m)$, Clearly $r > 0$

and for $r = \min d(x, y_m)$

The ball $B(x,r)$ contains no point of S distinct from x. So x is not a limit point of S. (Contradiction)

Theorm 2.7 : In a metric space X, a set is closed iff S contains all it's limit points.

proof : Suppose S is closed and x be limit point of S. Assume $x \notin S$

since $X \setminus S$ is open set, $\exists$ a ball $B(x,r) \subset X \setminus S$

(x is interior point of $X \setminus S$)

$\sim(\forall r > 0,\ B(x,r)$ has an elemet in $X \setminus S)$

$\therefore$ x is not a limit point of S (contradiction)

So $x \in S$ $\therefore$ S is closed $\Rightarrow$ S contains all it's limit points.

Suppose S contains all its limit points.

Let $x \in X \setminus S$ so x is not a limit point of S. Therefore

$\sim(\exists r$ s.t $\forall B(x,r)$ has elet in $S)$

$\exists$ an open ball $B(x,r) \subseteq X \setminus S$

So $X \setminus S$ is an open set.

$\therefore$ S is a closed set.

Exe: every real number is a limit point of the set $\mathbb{Q}$.

let

$$S_r = \{ x \in \mathbb{Q} \mid x < r, \text{ for some } r \}$$

now, $\forall R > 0$, $B(r, R) = \{ y \in \mathbb{R} \mid d(y, r) < R \}$

$\| y - r \| < R$

$| r - y | < R$

so for $y_0 = r - \frac{R}{2} \in B(r, R)$

$y_0 \in \mathbb{R}$

then $y_0 < r$

$\Rightarrow y_0 < q < r$

so $q \in B(r, R)$

and $q \in S_r$

so $\forall R > 0$, there is an element in $B(r, R)$ $q \in S_r$.

$\therefore r$ is a limit point.

8^{th} Aug :

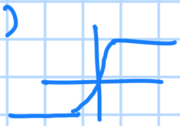
Def^n: (i) The number of element in a finite set S is called its cardinality.

(ii) A set S is said to be countably infinite if $\exists$ a bijective function $f: \mathbb{N} \to S$

(iii) An infinite set, which is not countable is called uncountable.

$f: \mathbb{R} \to (0,1)$

$\tan^{-1}(x)$



$$f(x) = \left(\tan^{-1}(x) + \frac{\pi}{2}\right)\frac{1}{\pi}$$

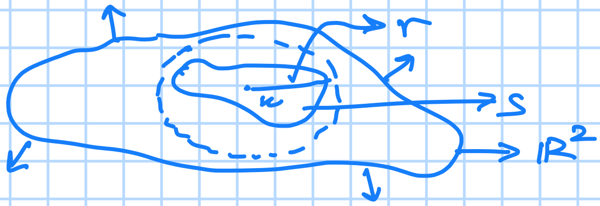
$$\underbrace{\underbrace{\left(\tan^{-1}(x) + \frac{\pi}{2}\right)}_{(0,\pi)}\frac{1}{\pi}}_{(0,1)}$$

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Revision: · $\mathbb{R}^n$, metric space

· Open space, closed space, limit points.

Def^n : Bounded : A set S in $\mathbb{R}^n$ is said to be **bounded** if $\exists\, x \in \mathbb{R}^n$ and $r > 0$ s.t $S \subseteq B(x, r)$



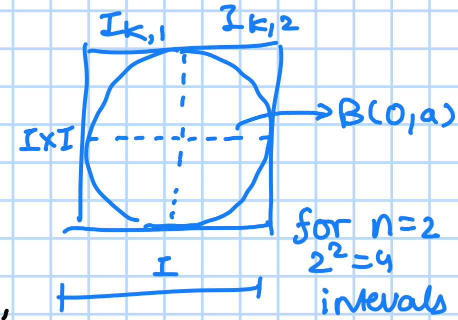
Note if $S \subseteq B(x, r)$ as $B(x,r)$ for $r' > \|x - 0\|$ $S \subseteq B(0, r')$

Theorm 2.9 : (Bolzano-Weierstrass theorm)
If S is a bounded set in $\mathbb{R}^n$ containg infinity many points, then $\exists$ at least one point in $\mathbb{R}^n$ which is a limit point of S.

proof : Since S is bounded, $\exists\, a > 0$ s.t $S \subseteq B(0, a)$
Therefore S is contained in a n-dimensional interval J_1 defined as the cartesian product

$$J_1 = I_1^{(1)} \times I_2^{(1)} \times \cdots \cdots I_n^{(1)}$$

where $I_k = [-a, a]$ $\forall\, k = 1, 2, \ldots n$



for $n=2$
$2^2 = 4$
intervals

each $I_k^{(1)}$ can be bisected to form two intervals, namely

$$I_{k,1} = [-a, 0] \text{ and } I_{k,2} = [0, a]$$

we consider all possible cartesian products of the form

$$I_{1,k_1} \times I_{2,k_2} \times \cdots\cdots I_{n,k_n} \quad \text{where } k_i = 1 \text{ or } 2$$

there are 2^n such n dimensional intervals and the union of all these n-dim intervals is J_1.

Because J_1 contains S, atleast one of the 2^n, n-dim intervals contains infinity many points of S.

One of this intervals which contains infinitely many points of S is deoted by J_2 which can be expressed as:

$$J_2 = I_1^{(2)} \times I_2^{(2)} \times \cdots\cdots I_n^{(2)}$$

where each I_k is one of the subintervals of I_k' of length a.

We now proceed with J_2 as we did for J_1, Bisecting each interval I_k and arriving at an n-dimensional interval J_3 containing infinitely many points from S.

Continuing this way, we obtain intervals $J_1, J_2, \ldots\ldots$ where $\forall m \in \mathbb{N}$, set J_m has the property that is contains inf. points of S and can be expressed as:

$$J_m = I_1^{(m)} \times I_2^{(m)} \times \cdots\cdots I_n^{(m)}$$

$\text{len}([-a, a]) = 2a = \dfrac{a}{2^{1-2}}$

Note that $I_k^{(m)} \subset I_k^{(1)}$ and we write $\text{len}([-a, 0)) = a = \dfrac{a}{2^{2-2}}$

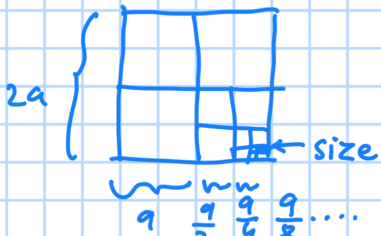
$$I_k^{(m)} = [a_k^m, b_k^m]$$

so $\text{len}([a_k^m, b_k^m]) = \dfrac{a}{2^{m-2}}$

$$I_1^{(m)} = [a_1^m, b_1^m]$$

$$[-a, a] \text{ for } I_1^{(1)} \; (\text{length} = 2a = \tfrac{a}{2^{1-2}})$$

Note: $b_k^m - a_k^m = \dfrac{a}{2^{m-2}}, \; \forall k = 1, 2, \ldots\ldots, \; \forall m \in \mathbb{N}$



$m=1$	$2a$	$= a/(2^{1-2})$
$m=2$	a	$= a/(2^{2-2})$
$m=3$	$\frac{a}{2}$	$= a/(2^{3-2})$
$\vdots$		$\vdots$

1) **Claim**: For each k, the sup of all left endpoints $a_k^{(m)}$ should be equal to the inf of all right endpoints $b_k^{(m)}$.

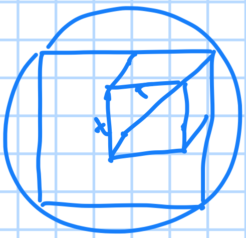
(Existance of sup and inf follows from completness axiom)

↳ Bounded above and below

(**Exercise**): let us determine the common value by t_k.

2) **Claim**: $t = (t_1, t_2, \ldots\ldots t_n)$ is a limit point of S.

Proof of Claim 2: Consider any n-ball $B(t, r)$ for some $r > 0$. The point $t \in J_m \; \forall m \in \mathbb{N}$ and so, if m_0 is s.t $\dfrac{a}{2^{m_0-2}} < \dfrac{r}{\sqrt{n}}$ then



$J_{m_0} = l \times l$

$l < \dfrac{r}{\sqrt{2}}$

$B(t, r) \supseteq J_{m_0}$. But since J_{m_0} contains infinitely many points of S, so will $B(t, r)$.

This proves that t is a limit point of S.

(Exercise): let us determine the common value by t_k.

Now as $I_k^{(m)} = [a_k^m, b_k^m]$ and

$$a_k^m - b_k^m = \frac{a}{2^{m-2}}$$

$$\sup\{a_1^1, a_1^2, a_1^3, \ldots\ldots\} = t_1$$

$$\sup\{a_2^1, a_2^2, a_2^3, \ldots\} = t_2$$

$$\vdots$$

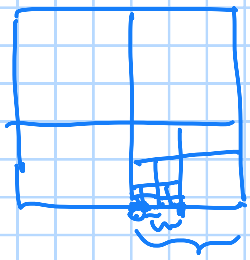
now, $t_i' = \inf\{b_1', b_2', b_3' \ldots\ldots\}$

t_1 is s.t $\forall i \in \mathbb{N}$

$$a_1^i \leq t_1$$

and t_1' is s.t $\forall i \in \mathbb{N}$

$$b_1^i \geq t_1'$$



if $t_1 \neq t_2$ then $\sup\{a_1^1, a_1^2, a_1^3 \ldots\} = t_1$
$\inf\{b_1', b_1^2, b_1^3 \ldots\} = t_1'$

monotonic and bounded above
so sup = limit

as limit a_1^n = limit b_1^n

$$\Rightarrow t_1 = t_1'$$

$$\therefore \sup\{a_1^n\} = \inf\{b_1^n\}$$

12th Aug : * open & closed sets in metric space. limit pts
* Bolzano-Weierstrass theorm

Defn : If (X,d) is a metric space and $S \subset X$, then S' denotes the set of all limit points of S.

The set $\bar{S} = S \cup S'$ is called closure of S.

Example :
(a) $[0,1]$ is closure of $(0,1]$

(b) $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is closure of $\{\frac{1}{n} : n \in \mathbb{N}\}$

Theorm 2.9: (Cantor intersection theorm) → also see nested interval theorm (special case)

Let $\{A_1, A_2, A_3, \dots\}$ be a countable collection of non-empty subsets of $\mathbb{R}^n$ s.t

i) $A_{k+1} \subseteq A_k \quad \forall k \in \mathbb{N}$ (Nested)

ii) Each set A_k is closed and A_1 is bounded (then all A_k are bounded)

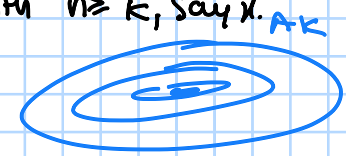
then the intersection $\bigcap_{k=1}^{\infty} A_k$ is closed and non-empty.

proof : let $S = \bigcap_{k=1}^{\infty} A_k$. Here S is closed (from coro 2.5)

we would next show that S is non-empty.

see proof

If any A_k contains only finitely many pts, then clearly one of those points should be in all A_n with $n \geq k$, say x.



as $A_{k+1} \subseteq A_k$

As there should be atleast one point as $A_{k+1} \subseteq A_k$ (nested) also each A_k is closed.

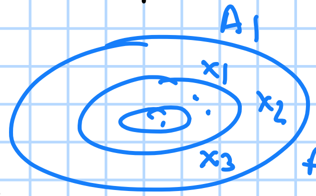
so, $\bigcap_{k=1}^{\infty} A_k \ni x$

Hence $\bigcap_{k=1}^{\infty} A_k$ is non empty.

Assume each A_k has infinetly many points.

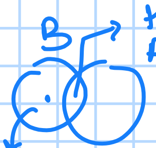
let $B = \{x_1, x_2, \dots\}$ be a collection of distinct points s.t $x_k \in A_k \; \forall k \in \mathbb{N}$

① B is bounded
② B has inf-many elements



Because B is an infinite set, By Bolzano-Weierstrass theorm, $\exists\, x \in \mathbb{R}^n$ s.t x is a limit point of B.

Each neighbourhood of x contains inf. many points of B. (Thm 2.6)



this A_k should have inf many points

neigh (x) contains inf many points

Since for each K, A_K contains all points of B except finitely many. Therefore each neighbourhood of x contains infinitely many points of A_K. i.e, x is a limit point of A_K, $\forall K \in \mathbb{N}$.

Since A_K are closed $\forall K \in \mathbb{N}$, we get $x \in A_K$, $\forall K \in \mathbb{N}$

Hence $x \in \bigcap_{K=1}^{\infty} A_K$, i.e $\bigcap A_K$ is non-empty.

<u>Example</u>: 1) $\{2^{-n} + 5^{-m} / n, m \in \mathbb{N}\}$

Limit points: 2^{-n} $\forall n \in \mathbb{N}$
5^{-m} $\forall m \in \mathbb{N}$
0

The set of all limit points of A is

$$\{2^{-n}, 5^{-m} \mid n, m \in \mathbb{N}\} \cup \{0\}$$

2) Take the decimal expansion of $\sqrt{2} = 1.r_1 r_2 r_3 \ldots$ Then the set $\{1, 1.r_1, 1.r_1 r_2 \ldots\}$

does not have a limit point in $\mathbb{Q}$
($\mathbb{Q}$ is not complete)

3) $A_K = [0, \frac{1}{K}]$, $K \in \mathbb{N}$
then $\bigcap_{K=1}^{\infty} A_K = \{0\}$

(as ① closed + bounded
② $A_{K+1} \subseteq A_K$)

4) $A_K = (0, \frac{1}{K})$, $K \in \mathbb{N}$, Then
$\bigcap_{K=1}^{\infty} A_K = \phi$

<u>Def^n</u>: A collection F of sets is said to be a <u>covering</u> of a given set S if

$$S \subseteq \bigcup_{A \in F} A$$

The collection F is said to <u>cover</u> S. If F is a collection of open sets, then we say F is <u>open covering</u>.

<u>Example</u>: $\{(\frac{1}{n}, \frac{2}{n}) \mid n = 2, 3, 4, \ldots\}$ is an open covering of $(0,1)$

$(0,1) \subseteq (\frac{1}{2}, 1) \cup (\frac{1}{3}, \frac{2}{3}) \cup (\frac{1}{4}, \frac{2}{4}) \cup \ldots\ldots$ is an open covg

13th Aug : * B.W theorem, Cantor int. theorem

* $A_n = [n, \infty)\ \forall n \in \mathbb{N}$, $\bigcap_{k=1}^{\infty} A_k = \phi$, ($A_n \neq \phi$, $A_{n+1} \subset A_n$, A_n closed)

But A_n is not bounded → this is a very important example

Examples cont. : $A_n = \left(0, \frac{1}{n}\right)$, $\forall n \in \mathbb{N}$

Open covering : Examples

① $\left\{ \left(\frac{1}{n}, \frac{2}{n}\right) \middle| n = 2, 3, \dots \right\}$ is an open covering for $(0,1)$

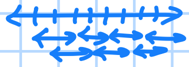
$\qquad\qquad F \qquad\qquad\qquad\qquad\qquad\qquad S$

as $S \subseteq \cup F$ and $\forall A$ in F is open

$\left(\frac{1}{2}, 1\right), \left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{4}, \frac{1}{2}\right) \dots$

② $\mathbb{R}$ is covered by $\{(a,b) \mid a, b \in \mathbb{R}\}$ which is not countable.

The countable set $\{(n, n+2) \mid n \in \mathbb{Z}\}$ is an open covering of $\mathbb{R}$



Theorem 2.10: Let $G = \{A_1, A_2, A_3, \dots\}$ denote the countable collection of n-balls of rational radii and centered at pt. with rational coordinates in $\mathbb{R}^n$. Assume $x \in \mathbb{R}^n$ and S be open set in $\mathbb{R}^n$ s.t $x \in S$. Then atleast one of the n-balls in G contains x and is subset of S.

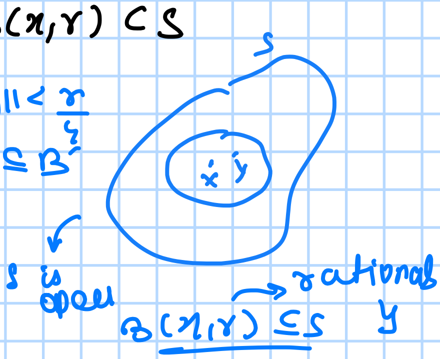
proof : Since S is open and $x \in S$, $\exists$ a n-ball $B(x, r) \subset S$

let $y = (y_1, y_2, \dots y_n)$ s.t

$y_k \in \mathbb{Q}$ and

$|x_i - y_i| < \frac{r}{4n}$, $\forall i = 1, 2, \dots n$ where

$x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$



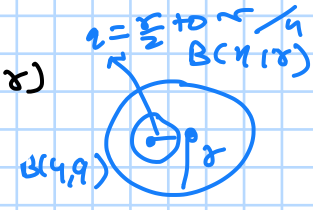
then, $\|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} < \sqrt{n\left(\frac{r^2}{(4n)^2}\right)} = \frac{r}{4\sqrt{n}} \leq \frac{r}{4}$

choose a rational number q s.t $\frac{r}{4} < q < \frac{r}{2}$

then $x \in B(y, q)$ and $B(y, q) \subseteq B(x, r)$

$\subseteq S$

$\left(\because d(x,y) < \frac{r}{4} < q\right)$



and for $z \in B(y,q)$ we have
$\forall z$ here $\|z-x\| \leq \|z-y\| + \|y-x\|$

$< q + \frac{r}{4} < r$ → see proof

$\therefore z \in B(x,r)$

AND $B(y,q) \in G$

Theorm 2.11 : (Lindelöf covering theorm)
Assume $A \subseteq \mathbb{R}^n$ and F be an open covering of A. Then $\exists$ a countable subcollection of F that covers A.

proof: let $G = \{A_1, A_2, \ldots\}$ be the countable collection of all n-balls with rational radii and centered at a pt with rational coordinates. Suppose $x \in A$,

$\exists$ a open set S in F s.t $x \in S$. By thm 2.10, $\exists$ an open ball $A_k \in G$ s.t $x \in A_k \subseteq S$.

There are many such A_k's but we choose only one of these which has smallest index, say $m(x)$, over all $S \in F$ s.t $S \in S$. Then for some $C \in F$, we have $x \in A_{m(x)} \subseteq C$. let

$H = \{A_{m(x)} \mid x \in A\}$ → countable collection which covers A.

we associate each set $A_{m(x)}$ of H one of the sets of F which contains $A_{m(x)}$ and we denote that set of F by $S_{m(x)}$

Hence, $\{S_{m(x)} \mid x \in A\}$ is a covering of A and is a countable subcollection of F. (as $x \in A_{m(x)} \subseteq C$)

19th Aug :

Theorm 2.12 : (Heine Borel theorm) Let A be a closed and bouded set in $\mathbb{R}^n$ and F be an open covering of A. Then $\exists$ a finite subcollection of F which coveres A. (also called subcovering)

proof: If F is a finite collection, then we are done.

Let F be not finite. By thm 2.11, $\exists$ a coutable subcovering of A wrt F, say $\{B_1, B_2, \ldots\}$.

Let $C_n = B_1 \cup B_2 \cup \ldots \cup B_n$

To show that : $\exists n_0 \in \mathbb{N}$ s.t $A \subseteq C_{n_0}$

Define coutable collection of closed sets $\{A_1, A_2, \ldots\}$

By $A_1 := A$ → closed

$A_k := A \cap C_k^c$ → closed as C_k^c is closed and A is closed

$\forall k > 1$

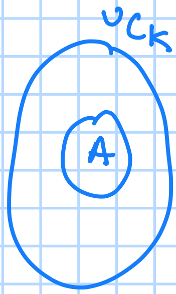
since $C_n \subseteq C_{n+1}$ $\forall n \in \mathbb{N}$

$\Rightarrow A_{n+1} \subseteq A_n$, $\forall n \in \mathbb{N}$

Note that A is bounded.

we would show that $\exists k_0 \in \mathbb{N}$ s.t $A_{k_0} = \phi$

$\Rightarrow A \subseteq C_{k_0}$



we have $\bigcap\limits_{k=1}^{\infty} A_k = \bigcap\limits_{k=1}^{\infty} (A \cap C_k^c) = A \cap \left(\bigcup\limits_{k=1}^{\infty} C_k\right)^c$

$= \phi$

$\left(\because \bigcup\limits_{k=1}^{\infty} C_k = \bigcup\limits_{k=1}^{\infty} B_k\right)$

But by cantor intersection theorm, $\exists k_0$ s.t

$A_{k_0} = \phi$ → see the theorm (negation)

ie $A \cap (C_{k_0})^c = \phi$

$\Rightarrow A \subseteq C_{k_0} = B_1 \cup B_2 \cup \cdots B_{k_0}$

Hence $(B_1, \ldots B_{k_0})$ is a finite subcovering of A.

Example : (i) $(0,1) \subseteq \bigcup\limits_{n=1}^{\infty} \left(0, 1-\frac{1}{n}\right)$

$(0,1)$ is bounded but not closed

(ii) $\mathbb{R} \subseteq \bigcup\limits_{n \in \mathbb{Z}} (n, n+2)$

$\mathbb{R}$ is not bounded

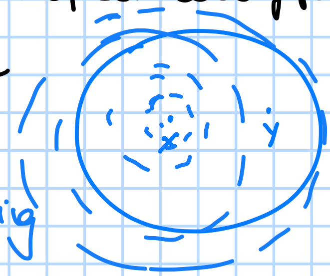
Def^n: A subset S of a metric space (X,d) is called **compact** if every open covering of S contains a finite subcovering.

Theorm 2.13: let S be a compact set in a metric space (X,d) Then:
(I) S is closed and bonded.
(II) Every infinte subset of S has a limit point in S.

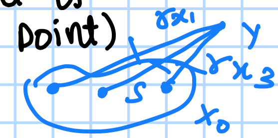
proof: (I) let $x \in S$. The collection $\{B(x,n) \mid n \in \mathbb{N}\}$ is an open coverig for S

Since S is compact, this covering contains a finite subcovering. Thus, $\exists n_0 \in \mathbb{N}$ s.t $B(x,n_0) \supseteq S$. Hence S is bounded.

choose the largest n corresponding to this finite subcovering.



To show that: S is closed (or S^c is open / every point is interior point)



let $y \in S^c$. let $r_x = \frac{1}{2} d(x,y)$, $\forall x \in S$ (clearly $r_x > 0$)

so, the collection $\{B(x,r_x) \mid x \in S\}$ is an open covering of S.

since S is compact, $\exists x_1, x_2, \ldots x_n \in S$ s.t $\bigcup_{k=1} B(x_k, r_{x_k}) \supseteq S$.

let $r := \min\{r_{x_1}, r_{x_2}, \ldots r_{x_n}\}$. clearly $r > 0$.

claim: $B(y,r) \cap S = \phi$

let $z \in B(y,r)$. Then $d(z,x_k) \geq d(y,x_k) - d(y,z)$
$= 2r_k - d(y,z) > 2r_k - r$
$\geq r_k$

$z \to x_k$ $\geq r_k$ $\therefore$ not ϕ $\therefore z \notin$ hence $\cap$

Thus $B(y,r) \cap \left(\bigcup_{k=1}^{n} B(x_k, r_{x_k})\right) = \phi$

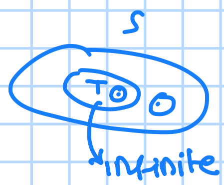
Hence $B(y,r) \subseteq S^c$ ($\because \{B(x_k, r_{x_k})\}$ is a covering of S)

now as $\exists r > 0$ s.t $B(y,r) \subseteq S^c$
$\Rightarrow \forall y \in S^c$ $B(y,r) \subseteq S^c$
$\therefore$ all points in S^c are interior points
$\Rightarrow S^c$ is open
$\Rightarrow$ <u>S is closed.</u>

20th Aug:

Theorem 2.13: Let S be a compact set in a metric space (X,d). Then:
(I) S is closed and bounded.
(II) Every infinite subset of S has a limit point in S.

Proof: (II) Let T be an infinite subset of S. Assume that T does not have a limit point in S. Then for each point of S, there exist an open ball centered at x such that either it does not contain any point of T or contains exactly one point for T (namely x itself).

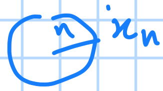


As x varies over S, the collection of such balls form an open covering of S. Since S is compact this covering has a finite subcovering of S.

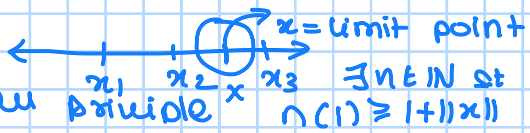
This finite subcovering also covers T. This implies, T is a finite set. This is a contradiction! Hence (ii) holds.

Theorem 2.14: Let $S \subseteq \mathbb{R}^n$, Then the following are equivalent.

(a) S is compact
(b) S is closed and bounded
(c) Every infinite subset T of S has a limit point in S.



Proof: (b) $\Rightarrow$ (a) is Heine-Borel theorem.
(a) $\Rightarrow$ (c) Follows from Thm 2.13 (II)
(c) $\Rightarrow$ (b) Assume that S is not bounded. So, for each $n \in \mathbb{N}$ $\exists\, x_n \in S$ s.t $\|x_n\| \geq n$. Since the set $\{x_1, x_2, \dots\}$ is an infinite subset of S, $\exists$ a limit point x of this subset in S.



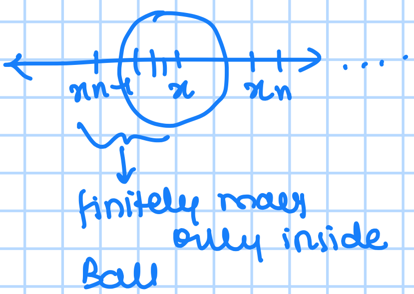
So for $n \geq 1 + \|x\|$ we have

$\|x\| \leq n-1 \qquad -\|x\| \geq 1-n$

$$\|x - x_n\| \geq \|x_n\| - \|x\| \geq n - \|x\| \geq n+1-n = 1$$

$\|x - x_n\| \geq 1$ → using triangular ineq.

Hence, there are atmost finitely many elements of $\{x_1, x_2, \dots\}$ in $B(x,1)$. This implies that x is not a limit point of $\{x_1, x_2, \dots\}$. $\divideontimes$



So S is bounded.

<u>S is closed proof</u>:

Let x be a limit point of S. Since there are inf. many points of S in $B(x, \frac{1}{k})$, $\forall k \in \mathbb{N}$, we choose a subset $\{x_1, x_2, \dots\}$ of distinct points of S such that $x_k \in B(x, \frac{1}{k})$, $\forall k \in \mathbb{N}$

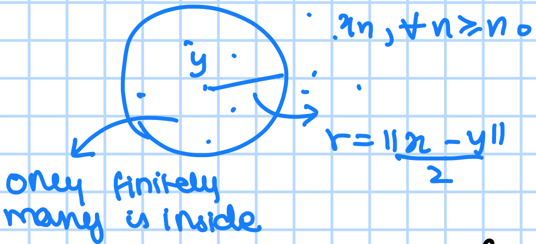
Clearly x is also a limit point of $T = \{x_1, x_2, \dots\}$.

Let $y \in \mathbb{R}^n$ be s.t $y \neq x$. Then:

$$\|x-y\| \leq \|x-x_n\| + \|x_n-y\| < \frac{1}{n} + \|x_n-y\|$$

also as $\|x-y\|>0$, $\exists n_0$ s.t $\frac{1}{n} < \frac{\|x-y\|}{2}$, $\forall n \geq n_0$ ↗ see: Archemedian property

So from $\|x-y\| < \frac{1}{n} + \|x_n-y\|$, for $n \geq n_0$,



$$\|x-y\| < \frac{\|x-y\|}{2} + \|x_n-y\|$$

$$\Rightarrow \frac{\|x-y\|}{2} < \|x_n-y\|$$

So y is not a limit point of T.

Thus by (c) we conclude $x \in S$.
$\therefore \forall$ limit point $\in S$
$\therefore$ S is closed.

Def^n: Two subsets A and B of a metric space (X,d) is said to be <u>seperated</u> if $A \cap \bar{B} = \phi$ and $\bar{A} \cap B = \phi$. A set $E \subseteq X$ is said to be <u>connected</u> if E is not a union of two non-empty separated sets.

<u>Example</u>: (I) $(0,1)$ and $(1,2)$ are seperated sets.

(II) $\underset{A}{[0,1]}$ and $\underset{B}{(1,2)}$ are not seperated sets.

$\bar{A} = [0,1]$
$\bar{B} = [1,2]$

$A \cap \bar{B} \neq \phi$ $\therefore$ <u>not-seperated</u>

($(0,2)$ is connected)

22nd Aug:

Theorem 2.15: Let Y be a closed subset of a compact metric space X. Then Y is compact.

Proof: Let $\{A_\alpha \mid \alpha \in I\}$ be an open covering for Y. Then $B = \{A_\alpha \mid \alpha \in I\} \cup Y^c$ is an open covering for X ($\because Y^c$ is open). Since X is compact, $\exists A_1, A_2, \dots A_n \in B$ s.t $\{A_1, A_2, \dots A_n, Y^c\}$ is a subcovering for X. (clearly $\{A_1, A_2, \dots A_n\}$ is a finite subcovering contained in B for Y) ($\because Y \cap Y^c = \phi$).

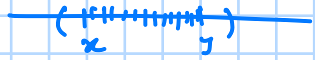
Hence Y is compact.

Example: For any set $E \subseteq \mathbb{R}$, $\sup E \in E$ → done, see down

Example: $Y = (0,1) \cup (1,2)$ is not connected

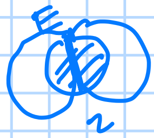
as $A = (0,1)$
$B = (1,2)$
$\bar{A} = [0,1]$
$\bar{B} = [1,2]$
$A \cap \bar{B} = \phi$
$\bar{A} \cap B = \phi$

Theorem 2.16: A subset E of $\mathbb{R}$ is connected iff the following holds: If $x, y \in E$ and $x < z < y$ for some $z \in \mathbb{R}$, then $z \in E$



Proof: Let $x, y \in E$ and assume $\exists z \in \mathbb{R}$ s.t $x < z < y$ and $z \notin E$.

Let $A_z = E \cap (-\infty, z)$ and
$B_z = E \cap (z, \infty)$



clearly $A_z \neq \phi$
$B_z \neq \phi$ $(\because x \in A_z, y \in B_z)$

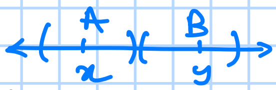
Also $\bar{A}_z \cap B_z = \phi$ $\left(\bar{A}_z \subseteq \overline{(-\infty, z)} = (-\infty, z] \text{ and } (-\infty, z] \cap B_z = \phi \right)$

Similarly
$\bar{B}_z \cap A_z = \phi$

Therefore E is not connected. $(\because A_z \cup B_z = E)$ which is a contradiction.

Hence $x, y \in E$ and $x < z < y$
$\Rightarrow z \in E$

Conversely, suppose E is not connected. Then $\exists$ non-empty seperated sets A and B s.t $A \cup B = E$.



gaps here
$(0,1)$ $(1,2)$

Let $x \in A$ and $y \in B$. Without loss of generality, let $x < y$. Define

$$z = \sup (A \cap [x, y])$$

clearly $z \in \bar{A}$, so $z \notin B$ $(\because \bar{A} \cap B = \phi)$ & $x \leq z < y$

if $z \notin A$, then $x < z < y$ and $z \notin A \cup B = E$

if $z \in A$, then z is not a limit point of B. So $\exists\, r>0$ s.t

$$(z-r, z+r) \cap B = \phi$$

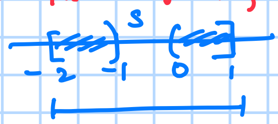
Choose $z_1 \in (z, z+r)$. Then $x \le z < z_1 < y$ and
$z_1 \notin A$ ($\because z = \sup(A \cap [x,y])$). Also $z_1 \notin B$.

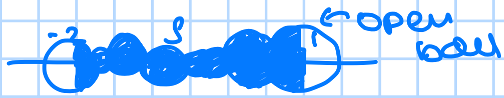
Hence $z_1 \notin A \cup B = E$

observe: A subset Y of a metric space (X,d) inherits a metric d' defined by $d'(x,y) = d(x,y)\ \forall\, x,y \in Y$.

observe: In a metric space $S = [-2,1]$ of $\mathbb{R}$, the set $[-2,-1) \cup (0,1]$ is open.

(open relative to S).





Note: The open ball of radius r centered at $y_0 \in Y$ in the (relative) metric of Y is $\{y \in Y \mid d(y_0, y) < r\} = B(y_0, r) \cap Y$

Note: E is open in S iff $E = S \cap G$ where G is open in X.

Example: For any set $E \subseteq \mathbb{R}$, $\sup E \in \bar{E}$

let $\sup E = s$, then $B(s, r) = (s-r, s+r)$ in $\mathbb{R}$.
now if $r>0$, then as $s-r<s$
$\exists\, x \in E$ s.t
$s-r < x \le s$

If $x = s$ then $\sup E \in E \Rightarrow \sup E \in \bar{E}$

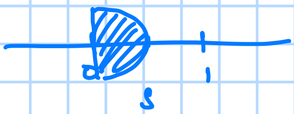
if $x < s$ then $x \in (s-r, s+r)$
$\therefore \forall\, r>0\ (B(s,r) - \{s\}) \cap E \neq \phi$
as x is one such element in it.
$\therefore$ s is a limit point of E
or
$s \in \bar{E}$

$\therefore \sup E \in \bar{E}$

Quiz-1
Tuesday in class
syllabus - till today

26th Aug : Read : Compactness, connectedness

Defn : Let S be a subset of a metric space (X,d). Then S is also a metric space w.r.t a metric $d' : S \times S \to \mathbb{R}$ s.t $d'(x,y) = d(x,y)\ \forall\ x,y \in S$.
This metric d' is called the relative metric induced by d on S and S is called metric subspace of X.



$B_S(0, \frac{1}{2}) = [0, \frac{1}{2})$

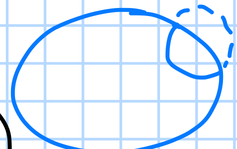
this is open

as $(-\frac{1}{2}, 0)$ does not exist in S.

Theorem 2.17 : Let (Y, d') be a metric subspace of (X,d). A subset E of Y is open relative to Y iff $E = Y \cap G$ where G is some open set in X.

proof : Let G be an open set in X and let $E = Y \cap G$

($\Leftarrow$) Let $x \in E$. So, $x \in G$ and hence $\exists$ an open ball $B(x,r)$ (w.r.t X) s.t $B(x,r) \subseteq G$ so,



open balls in relative metric

$Y \cap B(x,r) \subseteq Y \cap G$ $\left(\because B_Y(x,r) = B_X(x,r) \cap Y \right)$

$B'(x,r) \subseteq Y \cap G$

$\therefore$ x is an interior point of E w.r.t d'.

Thus E is open relative to Y.

($\Rightarrow$) Conversely let E be open relative to Y.
so, for every $x \in E$, $\exists r_x > 0$ s.t $B_X(x, r_x) \cap Y \subseteq E$

let $G = \bigcup_{x \in E} B(x, r_x)$. Then $E = Y \cap (\bigcup_{x \in E} B(x, r_x))$

i.e $E = Y \cap G$, where G is open X.

Theorem 2.18 : Let (Y, d') be a metric subspace of (X,d). A subset E of Y is closed relative to Y iff $E = Y \cap C$ where C is in some closed set in X.

proof : ($\Leftarrow$) $E = Y \cap C$, now if C is closed, $\forall x \in C$
$(B_x(x,r) - \{x\}) \cap C \neq \phi$
and also as $Y \cap C \neq \phi$ (if ϕ the closed)
then there will be inf. many points in $Y \cap C$ all limit points. $\therefore E = Y \cap C$ is closed.

($\Rightarrow$) E is closed, then $\forall r > 0$
$(B_X(x,r) - \{x\}) \cap E \neq \phi$
$\Rightarrow (B_X(x,r) \cap Y - \{x\} \cap Y) \cap E \neq \phi$
$\Rightarrow (B_Y(x,r) - \{x\}) \cap E \neq \phi$
if C is collection of all points s.t this happens, then C is closed
and
$E = Y \cap C$

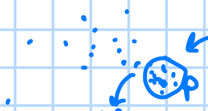
Def^n: Let S be a subset of a metric space X. A pt $x \in X$ is said to be a boundary pt of S if every ball $B(x,r)$ contains atleast one pt. of S and one pt of S^c. The set of all boundary pts of S is called the boundary of S and is denoted by ∂S.

Example: (I) $\partial(0,1) = \{0,1\}$
(II) $\partial[0,1] = \{0,1\}$
(III) $\partial \mathbb{Q} = \mathbb{R}$ $\longrightarrow$ this is because any point in $\mathbb{R}$ will have a ball in $\mathbb{Q}$ and $\mathbb{Q}^c$.

3. Sequences and continuity

Def^n: A seq. $\{x_n\}_{n=1}^{\infty}$ of points in a metric space (X,d) is said to converge if $\exists$ a pt. $p \in X$ with the following property:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t } d(p, x_n) < \varepsilon, \forall n \geq n_0$$



After a stage $\forall n \geq n_0$ will be inside this ball

eg: $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

$B(0, 0.01) = (-0.01, 0.01)$ After a stage they will be in this ball.

In this we say $\{x_n\}_{n=1}^{\infty}$ converges to p and we write $x_n \to p$ as $n \to \infty$

If $\exists$ no such p, then $\{x_n\}_{n=1}^{\infty}$ is said to diverge.

Eg. (I) $\lim_{n\to\infty} \frac{1}{n} = 0$ (For $\varepsilon > 0$, by Archemedian property $\exists n_0 \in \mathbb{N}$ s.t $n_0 \varepsilon > 1$ so $\frac{1}{n_0} < \varepsilon$ $\forall n \geq n_0$ i.e $\left|0 - \frac{1}{n}\right| < \varepsilon$ $\forall n \geq n_0$)

($\forall r \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t $\forall n > N \Rightarrow a_n > r$ then $a_n \to \infty$)

(II) $\{n^2\}_{n=1}^{\infty}$ diverges ($\because \varepsilon = 0.1$)

(III) Any constant sequence $\{x_n\}_{n=1}^{\infty}$ (where $x_n = c$) converges to c.

(IV) $\{(-1)^n\}_{n=1}^{\infty}$ diverges ($\because \varepsilon = 0.1$) (if cowg then $|(-1)^{2n} - L| < 1 \Rightarrow L \in (0,2)$ and $|(-1)^{2n+1} - L| < 1 \Rightarrow L \in (-2,0)$)

We do the following in $\mathbb{R}$ but they are true in any metric space.

Prop 3.1: (I) A convergent sequence has a unique limit point.

(II) A convergent sequence is bounded.

proof:
(I) Assume that seq $\{a_n\}_{n=1}^{\infty}$ is s.t $a_n \to p$ and $a_n \to q$ as $n \to \infty$.

Let $\varepsilon = |p - q|$. So, $\exists n_1 \in \mathbb{N}$ s.t $\forall n \geq n_1$ $|a_n - p| < \varepsilon/2$

also $\exists n_2 \in \mathbb{N}$ s.t $\forall n \geq n_2$, $|a_n - q| < \varepsilon/2$

now $|p - q| = |p - a_n + a_n - q| \leq |p - a_n| + |a_n - q| < \varepsilon/2 + \varepsilon/2 = \varepsilon = |p - q|$, $\forall n \geq \max\{n_1, n_2\}$

but $|p - q| < |p - q|$ $*$

$\therefore p = q$

(11) let $\{a_n\}_{n=1}^{\infty}$ be a seq s.t $a_n \to a$ as $n \to \infty$

so $\exists n_0 \in \mathbb{N}$ s.t $|a - a_n| < 1$, $\forall n \geq n_0$

Then

$$|a_n| \leq |a| + |a_n - a| < |a| + 1$$

$$\Rightarrow |a_n| < |a| + 1 \quad , \forall n \geq n_0$$

let $\alpha := \max\{|a_1|, |a_2|, \ldots\ldots |a_{n_0-1}|, |a|+1\}$

Then

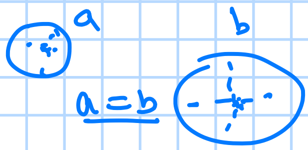
$$|a_n| \leq \alpha \quad , \forall n \in \mathbb{N}$$

$\therefore \{a_n\}_{n=1}^{\infty}$ is bounded.

27th Aug:

Defn: A sequence is a function f with domains as $\mathbb{N}$ and write $\{x_n\}_{n=1}^{\infty}$ where $x_n := f(n)$, $\forall n \in \mathbb{N}$

Note: ① A seq if has a limit then it is unique
② A seq if has a limit then it is bounded



Note: the above two are true for any matric space.
$|x-y| = d(x,y)$

Prop 3.2: In $\mathbb{R}$ (or $\mathbb{C}$) let $\{a_n\}_{a_n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be s.t $a_n \to a$, $b_n \to b$ as $n \to \infty$

(i) $a_n + b_n \to a+b$ as $n \to \infty$
(ii) $ra_n \to ra$ as $n \to \infty$ for some $r \in \mathbb{R}$ (or $\mathbb{C}$)
(iii) $a_n b_n \to ab$ as $n \to \infty$
(iv) If $a \neq 0$, $\exists m \in \mathbb{N}$ s.t $a_n \neq 0$, $\forall n \geq m$ and $\frac{1}{a_n} \to \frac{1}{a}$ as $n \to \infty$

proof: let $\varepsilon > 0$, so $\exists n_1, n_2 \in \mathbb{N}$ s.t $|a_n - a| < \varepsilon$ $\forall n \geq n_1$ and $|b_n - b| < \varepsilon$ $\forall n \geq n_2$.

(i) $|(a_n+b_n)-(a+b)| < |a_n - a| + |b_n - b| \leq \varepsilon + \varepsilon = 2\varepsilon$ ← this is fine
for $n \geq n_0 = \max\{n_1, n_2\}$

Hence $a_n + b_n \to a+b$ as $n \to \infty$

(ii) $|ra_n - ra| = |r||a_n - a| < |r|\varepsilon$ $\forall n \geq n_1$
hence $ra_n \to ra$ as $n \to \infty$

(iii) $$|a_nb_n - ab| = |a_nb_n - a_nb + a_nb - ab|$$
$$= |a_n(b_n - b) + b(a_n - a)|$$
$$\leq |a_n||b_n - b| + |b||a_n - a|$$
Note: since $a_n \to a$ as $n \to \infty$, by Prop 3.1 (ii), $\exists \alpha$ s.t $|a_n| \leq \alpha$ $\forall n \in \mathbb{N}$

$$\leq \alpha\varepsilon + |b|\varepsilon = (\alpha + |b|)\varepsilon \quad \forall n \geq n_0 = \max\{n_1, n_2\}$$
Hence $a_nb_n \to ab$ as $n \to \infty$

(iv) Since $a_n \to a$, $\exists m \in \mathbb{N}$ s.t $|a_n - a| < \frac{|a|}{2}$ $\left(\because \varepsilon = \frac{|a|}{2}\right)$ $\forall n \geq n_0$
then $|a_n| \geq |a| - |a_n - a|$
$\Rightarrow |a_n| \geq |a| - |a_n - a| > |a| - \frac{|a|}{2}$
$\Rightarrow |a_n| > \frac{|a|}{2}$ $\Rightarrow$ $\frac{1}{|a_n|} < \frac{2}{|a|}$

so, $$\left|\frac{1}{a} - \frac{1}{a_n}\right| = \frac{|a_n - a|}{|a||a_n|} < \frac{\varepsilon \times 2}{|a||a|} = \frac{2\varepsilon}{|a|^2}$$ for $\forall n \geq n_0 = \max\{n_1, M\}$

hence $\frac{1}{a_n} \to \frac{1}{a}$ as $n \to \infty$

Prop 3.3: (Sandwich theorem) Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be sequences in $\mathbb{R}$ and $c \in \mathbb{R}$ be s.t

(i) $a_n \leq c_n \leq b_n$, $\forall\, n \in \mathbb{N}$

(ii) $a_n \to c$ and $b_n \to c$ as $n \to \infty$

then $c_n \to c$ as $n \to \infty$

Proof: Let $\varepsilon > 0$, since $a_n \to c$ as $n \to \infty$, $\exists\, n_1 \in \mathbb{N}$ s.t
$c - a_n < \varepsilon \quad \forall\, n \geq n_1$

since $b_n \to c$ as $n \to \infty$, $\exists\, n_2 \in \mathbb{N}$ s.t
$b_n - b < \varepsilon \quad \forall\, n \geq n_2$

Let $n_0 = \max\{n_1, n_2\}$, then for $n \geq n_0$

$$-\varepsilon < a_n - c < b_n - c < \varepsilon$$
$$\Rightarrow -\varepsilon < a_n - c \leq c_n - c \leq b_n - c < \varepsilon$$
$$\Rightarrow |c_n - c| < \varepsilon \quad \forall\, n \geq n_0 = \max\{n_1, n_2\}$$

Def^n: The set of all real numbers together with two symbols $+\infty, -\infty$ is called the set of extended real numbers $(\mathbb{R}^*)$ which satisfy the following properties:

(i) If $x \in \mathbb{R}$, then
$$x + (+\infty) = +\infty$$
$$x - (+\infty) = -\infty$$
$$x - (-\infty) = +\infty$$
$$x + (-\infty) = -\infty$$
$$\frac{x}{-\infty} = \frac{x}{+\infty} = 0$$

(ii) If $x > 0$, then
$$x(+\infty) = +\infty$$
$$x(-\infty) = -\infty$$

(iii) If $x < 0$, then
$$x(-\infty) = +\infty$$
$$x(+\infty) = -\infty$$

(iv) $+\infty + (+\infty) = (+\infty)(+\infty) = (-\infty)(-\infty) = +\infty$
$(-\infty) + (-\infty) = (-\infty)(+\infty) = (+\infty)(-\infty) = -\infty$

(v) If $x \in \mathbb{R}$, then $-\infty < x < \infty$

Note: (i) If a subset of $\mathbb{R}$ is not bounded above, then we say that the $\sup A = +\infty$

Similarly for not bounded below, $\inf A = -\infty$

(ii) $\mathbb{R}^*$ is not a field

Def^n: Given a sequence $\{x_n\}_{n=1}^{\infty}$, consider a sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers, such that $n_1 < n_2 < \cdots$ then the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is called a subsequence of $\{x_n\}_{n=1}^{\infty}$.

$\{x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4} \ldots\}$

eg: $\{x_1, x_3, x_7, x_{11}, x_{12}, x_{20}, \ldots\}$

Trivial subsequence: Sequence itself $\left(\{n_k\}_{k=1}^{\infty} = \{k\}_{k=1}^{\infty}\right)$

Remark : $\{x_n\}_{n=1}^{\infty}$ converges to p iff every subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to p. (Exercise) ← done

proof: ($\Leftarrow$) As every subsequence converges, for the trivial subsequence,

$$\{n_k\}_{k=1}^{\infty} = \{k\}_{k=1}^{\infty}$$

is the sequence itself.

$\therefore$ the seq converge to p.

($\Rightarrow$) now as $\{x_n\}_{n=1}^{\infty}$ converge to p,

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t

$$|x_n - p| < \varepsilon$$

now for a subseq $\{n_k\}_{k=1}^{\infty}$

let $n_{k_0} \geq n_0$

then

$$|x_{n_{k_0}} - p| < \varepsilon$$

also $\forall n_k > n_{k_0}$ this will be true

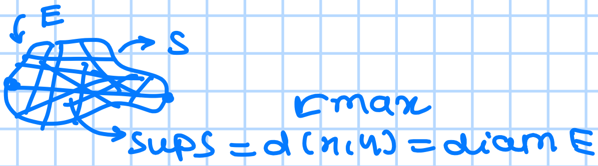
$\therefore \forall \varepsilon > 0 \ \exists k_0 \in \mathbb{N}$ s.t

$$|x_{n_k} - p| < \varepsilon \quad \forall k \geq k_0$$

$\therefore$ subsequences also converge to the same point p.

2nd sept : **Recap** : convergent seq, unique limit, bdd, sandwich theorem, $\mathbb{R}^*$ ext $\mathbb{R}$, and subsequences.

Defn : Let E be a subset of a metric space (X,d) and let $S = \{d(x,y) \mid x, y \in E\}$. Then the $\sup S$ is called the diameter of E, denoted by $\operatorname{diam} E$.



Theorem 3.4 : (a) If $\{x_n\}_{n=1}^{\infty}$ is a sequence of a compact metric space (X,d), then some subseq $\{x_{n_k}\}_{k=1}^{\infty}$ converges to a point in X.

(Bolzano-Weirstrass theorem for sequences)

(b) In $\mathbb{R}^n$, every bounded sequence has a convergent subsequence.
(from compl. axiom)

proof :

(a) Let $E = \{x_n \mid n \in \mathbb{N}\}$ be the range of $\{x_n\}_{n=1}^{\infty}$. If E is finite, $\exists x \in E$ and a seq $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ s.t $n_1 < n_2 < \dots$ s.t

$$x_{n_1} = x_{n_2} = x_{n_3} = \dots = x$$

the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x.

If E is an infinite set, then E has a limit point p in X because X is compact. (Prop 2.13)

Choose a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers s.t $x_{n_k} \in B(p, \frac{1}{k})$ $\forall k \in \mathbb{N}$

Then $\{x_{n_k}\}_{k=1}^{\infty}$ converges to p.

(b) Let $\{x_n\}_{n=1}^{\infty}$ be bounded so,

$\exists a \in \mathbb{R}^n$ and $r > 0$ s.t $\{x_n \mid n \in \mathbb{N}\} \subseteq B(a,r) \subseteq \overline{B}(a,r)$.
By Heine Borel theorem $\overline{B}(a,r)$ is compact.
(closed and bounded → finite subcovering)

Now as $\{x_n\}_{n=1}^{\infty}$ is a seq in a compact space,
by (a), $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Defn : A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X,d) is said to be a cauchy seq if for every $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t
$d(x_n, x_m) < \varepsilon$, $\forall n, m \geq n_0$

Remark : convergent sequence ⇒ cauchy sequence

$$\left(d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) \right)$$

the other direction may not always be true.

Eg : $(\mathbb{Q}, d)$ where d is the real metric.
$\{x_n\}_{n=1}^{\infty}$ be a seq in $\mathbb{Q}$ s.t $x_n \to \sqrt{2}$ as $n \to \infty$
the sequence is cauchy but not convergent in $(\mathbb{Q}, d)$

Theorm 3.5 : (a) let $\{x_n\}_{n=1}^{\infty}$ be a cauchy sequence in a compact metric space (X,d). Then $\{x_n\}_{n=1}^{\infty}$ converge to X.

(b) In $\mathbb{R}^n$, every cauchy sequence converges.
(from comp axiom)

Proof :
(a) let $\{x_n\}_{n=1}^{\infty}$ be a cauchy seq in a compact matric space (X,d).

suppose $A=\{x_n \mid n\in\mathbb{N}\}$ denote the range of $\{x_n\}_{n=1}^{\infty}$.

If A is finite, then $\{x_n\}_{n=1}^{\infty}$ is eventualy constant is an element of A.

Hence $\{x_n\}_{n=1}^{\infty}$ converges.

If A be infinite, then E has a limit point say P in X because X is compact by theorm 2.13.

we would show $x_n \to P$ as $n \to \infty$

given $\varepsilon>0$, $\exists n_0\in\mathbb{N}$ s.t $d(x_n,x_m)<\varepsilon/2$, $\forall n,m\geq n_0$
Since p is a limit point of A, $\exists m_0\in\mathbb{N}$, $m_0\geq n_0$ s.t

$$d(p,x_{m_0})<\varepsilon/2$$

Therefore for $n\geq n_0$, $d(x_n,P)\leq d(x_n,x_{m_0})+d(x_{m_0},p)<\varepsilon$
so $x_n\to p$ as $n\to\infty$.

3^{rd} Sept : Subsequences – in Compact has subseq conv
– in $\mathbb{R}^n$ has a conv subseq
Cauchy seq – in compact, cauchy $\Rightarrow$ conv

Theorem 3.5 : (a) Let $\{x_n\}_{n=1}^{\infty}$ be a cauchy sequence in a compact metric space (X,d). Then $\{x_n\}_{n=1}^{\infty}$ converge to X.

(b) In $\mathbb{R}^n$, every cauchy sequence converges.

proof (b) To show: Cauchy seq is bounded → see if conv seq are bounded in M or not.

← using diameter we can show this

Let $\{x_n\}_{n=1}^{\infty}$ be a cauchy seq in $\mathbb{R}^n$. Let $E_n = \{x_m \mid m \geq n\}$. Since $\{x_n\}_{n=1}^{\infty}$ is a cauchy, diam E_n converges to 0 as $n \to \infty$. So, (prove this)
$\exists$ $n_0 \in \mathbb{N}$ s.t diam $E_{n_0} < 1$.

The range of $\{x_n\}_{n=1}^{\infty}$ is $E_{n_0} \cup \{x_1, x_2, \dots x_{n_0-1}\}$
see proof
Hence $\{x_n\}_{n=1}^{\infty}$ is a bounded seq. By Heine-Borel thm any bounded set in $\mathbb{R}^n$ is contained in a compact set.

so by (a), the seq $\{x_n\}_{n=1}^{\infty}$ converges.

Defn : A metric space (X,d) is said to be complete if every Cauchy seq in X conv.
Eg : $\mathbb{R}^n$, compact metric space, $\mathbb{C}^n$, etc.

↳ compact metric space
$\Downarrow$
complete

but : complete $\not\Rightarrow$ compact metric space ($\mathbb{R}^n$ is example)
relative space $\Rightarrow$ where complete + bounded $\not\Rightarrow$ compact
+ infinite set

Defn: A seq in $\{x_n\}_{n=1}^{\infty}$ is said to be

(a) monotonically increasing if $x_n \leq x_{n+1}$, $\forall n \in \mathbb{N}$

(b) monotonically decreasing if $x_n \geq x_{n+1}$, $\forall n \in \mathbb{N}$

Theorem 3.6 : Suppose $\{x_n\}_{n=1}^{\infty}$ is a monotonically increasing seq in $\mathbb{R}$. Then $\{x_n\}_{n=1}^{\infty}$ is bounded above iff $\{x_n\}_{n=1}^{\infty}$ is convergent.

proof : ($\Leftarrow$) If $\{x_n\}_{n=1}^{\infty}$ is conv then $\{x_n\}_{n=1}^{\infty}$ is bounded, and hence bounded above.

($\Rightarrow$) Let $\{x_n\}_{n=1}^{\infty}$ be bounded above in $\mathbb{R}$ and is monotonically inc.

So, $s = \underset{<\infty}{\sup}\{x_n \mid n \in \mathbb{N}\}$ ($\because$ completeness axiom)

we have $x_n \leq s$, $\forall n \in \mathbb{N}$

Claim : x_n converges to s. ($|x_n - s| < \varepsilon, \forall n \geq n_0$)
Let $\varepsilon > 0$. So $\exists x_{n_0}$ s.t
$s - \varepsilon < x_{n_0} \leq s$ (by property of supremum)

Thus $s - \varepsilon < x_{n_0} \leq x_n \leq s$, $\forall n \geq n_0$ (as mon. inc) i.e $\quad s-\varepsilon < x_n < s+\varepsilon, \ \forall \ n \geq n_0$

$\Rightarrow \ -\varepsilon < x_n - s < \varepsilon, \ \forall \ n \geq n_0$

$\Rightarrow \ |x_n - s| < \varepsilon, \ \forall \ n \geq n_0$

$\therefore \ x_n$ convergest to s

Remarks: A monotonically decreasing seq which is bounded below is convergent (in $\mathbb{R}$)

$$\underbrace{\lim_{x \to p}}_{\text{we need a metric here}} \underbrace{f(x) = q}_{\text{we need a metric here too}}$$

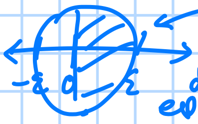
Def^n: let (X, d) and (Y, d') be metric spaces. suppose $E \subseteq X$, $f: E \to Y$ is a function and p is a limit point of E. Then we write $f(x) \to q$ as $x \to p$

Or $\lim_{x \to p} f(x) = q$

if $\exists \ q \in Y$ with the following property:

$\forall \varepsilon > 0, \ \exists \delta > 0$ s.t for $x \in E$

$0 < d(x, p) < \delta \Rightarrow d(f(x), q) < \varepsilon$



deleted epsilon ball

Example: $\lim_{x \to p} x^2 = p^2$

let $\varepsilon > 0$,

$|x^2 - p^2| = |x-p||x+p| \leq |x-p| \ (|x-p| + 2|p|)$

with $\delta = \min\left\{1, \frac{\varepsilon}{1+2|p|}\right\}$

if $0 < |x-p| < \delta$

then $|x-p| < \frac{\varepsilon}{1+2|p|}$

$|x-p| + 2|p| < 1 + 2|p|$

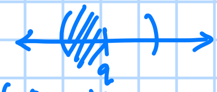
so $|x^2 - p^2| < \frac{\varepsilon}{\cancel{1+2|p|}} \times \cancel{(1+2|p|)} = \varepsilon$

4th sept :

seq conveg : $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t
$d(x, x_n) < \varepsilon \;\; \forall n \geq n_0$

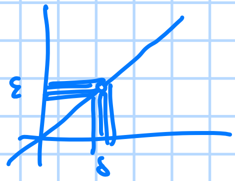
function : $f : E \to Y$, $E \subseteq X$, X, Y metric spaces. p is a limit point of E
then $\lim_{x \to p} f(x) = q$

if $\forall \varepsilon > 0, \exists \delta > 0$ s.t
$0 < d(x, p) < \delta \Rightarrow d'(f(x), q) < \varepsilon$



Examples:

① $f(x) = \begin{cases} 0 & \text{if } x = 1 \\ x & \text{otherwise} \end{cases}$ Allow $q \neq f(p)$



$\lim_{x \to 1} f(x) = 1$ but see $f(1) = 0$
$\therefore$ we are looking at the behaviour around that x.

② $g(x) = x \;\; \forall x \in \mathbb{R}$, $\lim_{x \to 1} g(x) = 1$

③ $h(x) = \begin{cases} 1 & \text{if } x = \frac{1}{4} \\ 0 & \text{otherwise} \end{cases}$, $h : \mathbb{R} \to \mathbb{R}$

Theorem 3.7 : let X, Y, E, f and p be as in the prev. Defn
Then we say $\lim_{x \to p} f(x) = q$ iff for every $\{x_n\}_{n=1}^{\infty}$ in E s.t
$\lim_{n \to \infty} x_n = p$, $x_n \neq p$ we get $\lim_{n \to \infty} f(x_n) = q$

proof :
($\Rightarrow$) Suppose $\lim_{x \to p} f(x) = q$ T.s.t $f(x_n) \to q$
Given $\varepsilon > 0$, $\exists n_0$ s.t $d'(f(x_n), q) < \varepsilon$

let $\varepsilon > 0$, then $\exists \delta > 0$ s.t ($\#$)
$0 < d(x, p) < \delta$
$\Rightarrow d'(f(x), q) < \varepsilon$ whenever $x \in E$.

Since $\lim_{n \to \infty} x_n = p$, $\exists n_0 \in \mathbb{N}$ s.t $\forall n \geq n_0$
$d(x_n, p) < \delta$
so by ($\#$) $d'(x_n, q) < \varepsilon \;\; \forall \; n \geq n_0$

but this means $\lim_{n \to \infty} f(x_n) = q$

($\Leftarrow$) Conversely, assume that $\lim_{x \to p} f(x) \neq q$

$\left(\begin{array}{l} \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t } 0 < d(x, p) < \delta \Rightarrow d'(f(x), q) < \varepsilon \\ \text{fails} \\ \text{for some } \varepsilon > 0 \text{ s.t } \forall \delta > 0, \exists x \in E \\ d'(f(x), q) \geq \varepsilon \end{array} \right)$

so for some $\varepsilon > 0$, we have $\forall \delta > 0$, $\exists x \in E$ s.t $d(x, p) < \delta$
but $d'(f(x), q) \geq \varepsilon$

fix this ε, Thus for $\delta = \frac{1}{n}$, $\exists x_n \in E$ s.t $0 < d(x_n, p) < \delta = \frac{1}{n}$
but $d'(f(x_n), p) \geq \varepsilon$

so for this choice of $\{x_n\}_{n=1}^{\infty}$, $x_n \in E$ we get $\lim_{n \to \infty} x_n = p$, $x_n \neq p$
and $\lim_{n \to \infty} f(x_n) \neq q$

This proves the theorem.

Some notations

Let (X,d) be a metric space and let $A \subseteq X$. Consider two functions $f: A \to \mathbb{C}$ and $g: A \to \mathbb{C}$.

The sum $f+g$ is defined as the function whose value at x is $f(x)+g(x)$, i.e

$$(f+g)(x) = f(x)+g(x)$$

Similarly difference $f-g$ product fg and quotient f/g is defined as

$$(f-g)(x) = f(x)-g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$(f/g)(x) = f(x)/g(x)$$

except that f/g is defined only at points x for which $g(x) \neq 0$

Theorem 3.8: Let $A \subseteq X$ where X is a metric space. Let p be a limit point of A and $f, g : A \to \mathbb{C}$, suppose

$$\lim_{x \to p} f(x) = a, \quad \lim_{x \to p} g(x) = b$$

then (a) $\lim_{x \to p} (f \pm g)(x) = a \pm b$

(b) $\lim_{x \to p} (f \cdot g)(x) = a \cdot b$

(c) $\lim_{x \to p} (f/g)(x) = a/b$ if $b \neq 0$

combining the sequences theorem this follows

proof: For $\forall \{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ s.t $x_n \neq p$, $y_n \neq p$ and $x_n \to p$ and $y_n \to p$ as $n \to \infty$, we get $f(x_n) \to a$, $g(x_n) \to b$ as $n \to \infty$ (by theorem 3.7)

see

The assertion follows using theorem 3.2

5th sept :

Theorm 3.8 : let $A \subseteq X$ where X is a metric space. let p be a limit point of A and $f, g : A \to \mathbb{C}$, suppose
$$\lim_{x \to p} f(x) = a, \quad \lim_{x \to p} g(x) = b$$
then (a) $\lim_{x \to p} (f \pm g)(x) = a \pm b$

combing the sequence theorm this follows

(b) $\lim_{x \to p} (f \cdot g)(x) = a \cdot b$

(c) $\lim_{x \to p} (f/g)(x) = a/b$ if $b \neq 0$

proof : For $\forall \{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ s.t $x_n \neq p$, $y_n \neq p$ and $x_n \to p$ and $y_n \to p$ as $n \to \infty$, we get $f(x_n) \to a$, $g(x_n) \to b$ as $n \to \infty$ (by theorm 3.7)

The assertion follows using theorm 3.2

that $\lim_{n \to \infty} f(x_n) \pm g(x_n) = a \pm b$ (theorm 3.2)

then $\lim_{x \to p} f(x) \pm g(x) = a \pm b$

(b), (c) follow similarly.

See :
$$\begin{tikzcd} (X,d) \arrow[r, "f"] & (Y,d') \arrow[r] \arrow[dr] & \mathbb{R}/\mathbb{C} \\ & & \mathbb{R}^n \end{tikzcd}$$

Remark - If f and g maps $A \subseteq X$ to $\mathbb{R}^n$, then (a) holds. (b) is replaced by (b') $\lim_{x \to p} (f \cdot g)(x) = a \cdot b$ (inner product)

Def^n : let (X,d) and (Y,d') be metric spaces and $f : X \to Y$ be a function. The function is said to be continuous at a point $p \in X$ if $\forall \varepsilon > 0, \exists \delta > 0$
$$\text{s.t } \underbrace{d(x,p) < \delta}_{\text{see here it is not a half ball}} \Rightarrow d(f(x), f(p)) < \varepsilon$$

If f is continuous at every point $p \in A \subseteq X$, then we say f is cont on A.

Note : If f is cont at p, then $\lim_{x \to p} f(x) = f(p)$

Theorm 3.9 : If $f : X \to Y$ is a function bet^n metric spaces and $p \in X$. Then f is cont at p iff for every seq $\{x_n\}_{n=1}^{\infty}$ in X convergent to p, the seq $\{f(x_n)\}_{n=1}^{\infty}$ in Y converges to $f(p)$, ie $\lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} x_n)$

proof : Similar to theorm 3.7 except that the condition $x_n \neq p$ is not required.

Eg : (1) polynomials are continious functions

(2) g/f when g and f are polynomials and f does not vanish, is cont.

(3) e^x, trigonometric functions, logarithm are cont (we will see later)

Theorm 3.10: Let X, Y and Z be metric sps. Let $f: X \to Y$ and $g: Y \to Z$ be functions, and let $h: X \to Z$ be a composite function defined by $h(x) = g(f(x))$ $\forall x \in X$

If f is cont at p and g is cont at $f(p)$, then h is cont at p.

Proof: Let $b = f(p)$. Let $\varepsilon > 0$, so $\exists \delta > 0$ s.t $d_Y(y, b) < \delta \Rightarrow d_Z(g(y), g(b)) < \varepsilon$ ($\because g$ is cont)

For this δ, $\exists \delta' > 0$ s.t

$$d_X(x, p) < \delta' \Rightarrow d_Y(f(x), f(p) = b) < \delta$$

($\because f$ is cont)

Combining these two statements and taking $y = f(x)$, we observe that as

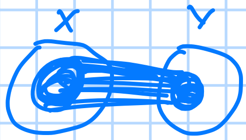
$$d_X(x, p) < \delta' \Rightarrow d_Z(h(x), h(p)) < \varepsilon \quad \left(\because h(x) = g(f(x)) \text{ and } h(p) = g(b)\right)$$

Hence h is continuous.

Theorm 3.11: If f and g are complex valued functions which are continious at a point p in metric space (X, d), then $f \pm g$, fg and f/g are each cont at p (except f/g should be considered for $g(p) \neq 0$)

Proof: The result is trivial if p is an isolated point of X. At limit points, the statement follows from theorm 3.8.

Def^n: Suppose $f: S \to T$ and $Y \subseteq T$. Then the inverse image of Y, denoted by $f^{-1}(Y)$ is defined as $\{x \in S \mid f(x) \in Y\}$



Note: (I) If $X = f^{-1}(Y)$ then $f(X) \subseteq Y$ (Equality for surjective f^n)

(II) If $Y = f(X)$, $X \subseteq f^{-1}(Y)$ (Equality for injective f^n)

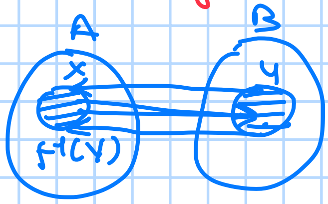
9th sept -

Notion of open sets is important when it comes to continuity

Remark: (I) If $X = f^{-1}(Y)$, then $f(x) \subseteq Y$

(II) If $Y = f(X)$, then $X \subseteq f^{-1}(Y)$ $\quad x \subseteq f^{-1}f(X)$

done proof $\quad f f^{-1}(Y) \subseteq Y$



Theorm 3.12: Suppose $f: X \to Y$ is a function where X and Y are metric spaces then f is cont on X iff "$f^{-1}(V)$ is open whenever V is open set"

proof:

($\Rightarrow$) let f be continuous. let V be an open set. T.s.t $f^{-1}(V)$ is open.

let $p \in f^{-1}(V)$, it is enough to show that p is an interior point $\left(B(p, \delta) \subseteq f^{-1}(V) \right)$

let $y = f(p)$. Since y is an interior point of V, $\exists\ \varepsilon > 0$ s.t $B(y, \varepsilon) \subseteq V$

Remark: f is cont $\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t $f(B(p, \delta)) \subseteq B(f(p), \varepsilon)$

since f is cont, $\exists\ \delta > 0$ s.t $f(B(p, \delta)) \subseteq B(y, \varepsilon)$

Now as $y = f(X)$ then $X \subseteq f^{-1}(Y)$

for $Y = f(B(p, \delta))$

then $B(p, \delta) \subseteq f^{-1} f(B(p, \delta)) \subseteq f^{-1}(B(y, \varepsilon))$

$\subseteq f^{-1}(V)$

$\therefore B(p, \delta) \subseteq f^{-1}(V)$

$\therefore p$ is an interior point of $f^{-1}(V)$, i.e $f^{-1}(V)$ is an open set

($\Leftarrow$) conversely let's assume that $f^{-1}(V)$ is open whenever V is open.

suppose $p \in X$. T.s.t f is cont at $p \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t $f(B(p, \delta)) \subseteq B(f(p), \varepsilon)$ (to show this)

let $\varepsilon > 0$. let $y = f(p)$. so, $f^{-1}(B(y, \varepsilon))$ is open

also $p \in f^{-1}(B(y, \varepsilon))$

as $p \in S$ and $S = f^{-1}(B(y, \varepsilon))$ is an interior point

$\therefore \exists\ \delta > 0$ s.t $B(p, \delta) \subseteq f^{-1}(B(y, \varepsilon))$

$\Rightarrow f(B(p, \delta)) \subseteq f f^{-1}(B(y, \varepsilon)) \subseteq B(y, \varepsilon)$ by remark ①

$\left(f f^{-1}(Y) \subseteq Y \right)$

$\therefore f(B(p, \delta)) \subseteq B(f(p), \varepsilon)$

$\therefore f$ is cont.

Remark: $f^{-1}(U^c) = (f^{-1}(U))^c \to$ done (show $f^{-1}(U) \cap f^{-1}(U^c) = \phi$ and $f^{-1}(U) \cup f^{-1}(U^c) = X$)

Theorm 3.13: Let $f: X \to Y$ where X and Y are metric spaces. f is cont iff $f^{-1}(U)$ is closed whenever U is closed.

proof:

($\Rightarrow$) f is cont, then let $U \subseteq Y$ be a closed set.
U^c is open set.
By theorm 3.12 $f^{-1}(U^c)$ is open
and $f^{-1}(U^c) = (f^{-1}(U))^c$ thus $(f^{-1}(U))^c$ is open
$\Rightarrow f^{-1}(U)$ is closed

($\Leftarrow$) conversely, $f^{-1}(U)$ is closed when U is closed, then t.s.t f is cont.
suppose V is an open set in Y, so V^c is closed. $f^{-1}(V^c)$ is closed by the hypothesis. Also $f^{-1}(V^c) = (f^{-1}(V))^c$ which is closed
$\Rightarrow f^{-1}(V)$ is open
$\therefore f^{-1}(V)$ and V are open
$\Rightarrow f$ is cont
($\because$ theorm 3.12)

Remark: (I) $f(A \cup B) = f(A) \cup f(B)$
(II) $f(A \cap B) \subseteq f(A) \cap f(B)$
(III) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
(IV) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
$\to$ done

Remark: (I) If $X = f^{-1}(Y)$, then $f(x) \subseteq Y$

(II) If $Y = f(X)$, then $X \subseteq f^{-1}(Y)$

proof: let $x \in X$ then
$\Rightarrow x \in f^{-1}(Y)$ $\leftarrow$ image of f
also as $x \in f^{-1}(Y)$
then $\exists y \in Y$ s.t
$f(x) \in Y$
$\Rightarrow f(X) \subseteq Y$

also, $Y = f(X)$ then $\forall x \in X$
$\to f(x) \in Y$
$\Rightarrow x \in f^{-1}(Y)$
$\Rightarrow X \subseteq f^{-1}(Y)$

Remark: (I) $f(A \cup B) = f(A) \cup f(B)$
(II) $f(A \cap B) \subseteq f(A) \cap f(B)$
(III) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
(IV) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

proof: (I) for $y \in f(A \cup B)$
$\exists x \in A \cup B$
s.t $f(x) = y$
i.e $x \in A \cup B$
$\Rightarrow x \in A$ or $x \in B$
$\Rightarrow f(x) \in f(A)$ or $f(B)$
$\Rightarrow f(x) \in f(A) \cup f(B)$
as $y = f(x) \in f(A) \cup f(B)$

$\forall y \in f(A \cup B)$
$\Rightarrow y \in f(A) \cup f(B)$

and if $y \in f(A) \cup f(B)$
then
$y \in f(A)$ or $y \in f(B)$
$\therefore\ \exists\, x$ s.t
$x \in A$ or $x \in B$
$\Rightarrow x \in A \cup B$
$\Rightarrow f(x) \in f(A \cup B)$
$\Rightarrow y \in f(A \cup B)$

$\therefore\ f(A \cup B) = f(A) \cup f(B)$

(ii) $f(A \cap B) \subseteq f(A) \cap f(B)$
for this
let $y \in f(A \cap B)$
then $\exists\, x$ s.t
$f(x) = y$
$x \in A \cap B$
so $x \in A$ and $x \in B$
$f(x) \in f(A)$ and $f(x) \in f(B)$
$\Rightarrow f(x) \in f(A) \cap f(B)$
$\Rightarrow y \in f(A) \cap f(B)$
$\Rightarrow f(A \cap B) \subseteq f(A) \cap f(B)$

(iii) $f^{-1}(A \cup B) = \{x \in \text{domain}(f) \text{ s.t } f(x) \in A \cup B\}$
$= \{x \in \text{dom}(f) \mid f(x) \in A \text{ or } f(x) \in B\}$
$= \{x \in \text{dom}(f) \mid f(x) \in A\} \cup \{x \in \text{dom}(f) \mid f(x) \in B\}$
$= f^{-1}(A) \cup f^{-1}(B)$

(iv) $f^{-1}(A \cap B) = \{x \in \text{dom}(f) \mid f(x) \in A \cap B\}$
$= \{x \in \text{dom}(f) \mid f(x) \in A \text{ and } f(x) \in B\}$
$= \{x \in \text{dom}(f) \mid f(x) \in A\} \cap \{x \in \text{dom}(f) \mid f(x) \in B\}$
$= f^{-1}(A) \cap f^{-1}(B)$

Remark: $f^{-1}(U^c) = (f^{-1}(U))^c$

Now, using the previous proved remark
$f^{-1}(U) \cap f^{-1}(U^c) = f^{-1}(U \cap U^c) = f^{-1}(\phi) = \phi$
$\Rightarrow f^{-1}(U) \cap f^{-1}(U^c) = \phi$
and $f^{-1}(U) \cup f^{-1}(U^c) = f^{-1}(U \cup U^c) = \underbrace{f^{-1}(Y) = X}_{\text{domain of } f}$

$\therefore\ f^{-1}(U) \cap f^{-1}(U^c) = \Phi$
and $f^{-1}(U) \cup f^{-1}(U^c) = X$
$\Rightarrow f^{-1}(U^c) = (f^{-1}(U))^c$

Theorm 3.14: Suppose $f: X \to Y$ be a function where X and Y are metric spaces. If f is continuous on a compact subset K of X, then $f(K)$ is compact subset of Y

10th Sept :

Recap: f is cont iff $f^{-1}(V)$ is open / closed whenever V is open / close

$$f(A\cup B) = f(A)\cup f(B)$$

Theorm 3.14 : Suppose $f: X \to Y$ be a function where X and Y are metric spaces. If f is continuous on a compact subset K of X, then $f(K)$ is compact subset of Y

K compact $\Rightarrow f(K)$ is compact

proof : Let F be an open covering of $f(K)$. Since f is continious $f^{-1}(A)\cap K$ are open in relative metric of K.

So $\{f^{-1}(A)\cap K \mid A\in F\}$ is an open covering for K.

Since K is compact, $\exists A_1, \ldots, A_n \in F$ s.t

$$K \subseteq \{f^{-1}(A_1)\cup f^{-1}(A_2)\cdots\cup f^{-1}(A_n)\}\cap K$$
$$\Rightarrow K \subseteq f^{-1}(A_1)\cup f^{-1}(A_2)\cdots\cup f^{-1}(A_n)$$
$$\Rightarrow f(K)\subseteq f(f^{-1}(A_1)\cup f^{-1}(A_2)\cdots\cup f^{-1}(A_n))$$
$$\Rightarrow f(K)\subseteq A_1\cup A_2\cup\cdots\cup A_n$$

$\therefore$ $f(K)$ is compact

Theorm 3.15 : Suppose $f: X\to \mathbb{R}$ is a function where X is a metric space. If f is a cont. on a compact subset K of X, then $\exists p, q \in X$ s.t

$$f(p) = \inf f(K)$$
$$f(q) = \sup f(K)$$

proof : Since f is continious, $f(K)$ is compact by theorm 3.14. and compact sets are closed and bounded (in $\mathbb{R}$).

(completness axiom used)

Since $f(K)$ is bounded, the supremum $f(K) < \infty$. since $f(K)$ is closed $\exists q\in K$ s.t $f(q) = \sup f(K)$ ($\because \sup f(K) \in \overline{f(K)}$ but $\overline{f(K)} = f(K)$)

Similarly $\exists p \in K$ s.t $f(p) = \inf f(K)$

Defn : Let $f: X\to Y$ be an injective function. Define inverse map

$$f^{-1}: f(X) \longrightarrow X \quad \text{by}$$
$$f^{-1}(y) = x \quad \text{if} \quad f(x) = y$$

(Previously we were seeing inverse image)

Here inverse image (sets) is different.

Theorm 3.16 : Assume X and Y are metric spaces and $f: X\to Y$ is a function. Suppose f is injective. If X is compact and f is continious on X then f^{-1} is continious on $f(X)$. (this concept is very important in topology – homeomorphism)

proof : T.s.t $(f^{-1})^{-1}(U)$ is closed whenever U is closed in X.
$\Rightarrow f(U)$ is closed whenever U is closed in X.

Suppose U is a closed set in X.
Since X is compact, U is also compact.
Since f is continious, $f(U)$ is compact. (Theorm 3.14)

This implies $f(U)$ is closed.
$\Rightarrow (f^{-1})^{-1}(U)$ is closed when U is closed
$\Rightarrow$ f^{-1} is continious

Remark: Such maps are important in "Topology."

Theorem 3.17: If f from X to Y is a continuous function where X, Y are metric spaces, and if E is a connected subset of X, then $f(E)$ is connected.

proof:

Let's assume that $f(E) = A \cup B$ where A and B are non-empty seperated sets. Let $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$. Then

$$E \subseteq f^{-1}f(E) = f^{-1}(A \cup B)$$
$$= f^{-1}(A) \cup f^{-1}(B)$$

so $E \subseteq G \cup H$

As $G = E \cap f^{-1}(A)$
$H = E \cap f^{-1}(B)$
$E \supseteq G \cup H$ $\left(G \cup H = E \cap (f^{-1}(A) \cup f^{-1}(B)) \subseteq E \right)$

So $E = G \cup H$

Also, $G \subseteq f^{-1}(A) \subseteq f^{-1}(\bar{A})$, since f is cont, inverse image of closed set is closed.

$\bar{G}$ is contained inside $f^{-1}(\bar{A})$

further $H \subseteq f^{-1}(\bar{B})$ closed
$\Rightarrow \bar{H} \subseteq f^{-1}(\bar{B})$

so $H \cap \bar{G} \subseteq f^{-1}(\bar{A}) \cap f^{-1}(B)$
$= f^{-1}(\underbrace{\bar{A} \cap B}_{\text{empty set}}) = \phi$

$\therefore H \cap \bar{G} = \phi$

similarly $\bar{H} \cap G = \phi$

clearly, H and G are non-empty. so, H and G are seperated sets.
This is a contradiction.
as $E = H \cup G$
connected

$\therefore F(E)$ is connected.

Midsem syllabus till today

12th sept -

Theorem 3.14 - Suppose $f: X \to Y$ be a function where X, Y are metric spaces. Let K be a compact subset of X and f be cont. Then $f(K)$ is compact in Y.

proof: Let F be an open covering of $f(K)$ in Y. Let $A \in F$, then "Is $f^{-1}(A)$ open?" Not necessarily. But $f|_K^{-1}(A) = (f^{-1}(A)) \cap K$ is open.

propn - Let $Y \subseteq X$, where X is a metric space, then Y is compact iff Y is compact in the relative metric space.

proof: Let Y be compact in X. Let $\{A_\alpha \mid \alpha \in I\}$ be an open covering of Y in
($\Rightarrow$) the relative metric of Y. So, $\exists$ open covering $\{B_\alpha \mid \alpha \in I\}$ of Y in X s.t
$$A_\alpha = B_\alpha \cap Y$$
since Y is compact in X, $\exists B_{\alpha_1}, \dots B_{\alpha_n} \in F$
s.t $\bigcup_{k=1}^{n} B_{\alpha_k} \supseteq Y$.
Hence $$\bigcup_{k=1}^{n} A_{\alpha_k} = \bigcup_{k=1}^{n} (B_{\alpha_k} \cap X) \supseteq Y$$

($\Leftarrow$) As Y is compact wrt X, for $Y \cap X$ there is a finite cover, then X^c, and the finite cover, covers Y, $\therefore$ finite subcover for Y
$\therefore Y$ is compact.

Theorem 3.18: (Intermediate Value theorem) Let $f: [a,b] \to \mathbb{R}$ be a continuous function and let $c \in \mathbb{R}$ be such that $f(a) < c < f(b)$ then $\exists x \in (a,b)$ s.t $f(x) = c$.

proof:
So, since $[a,b]$ is a connected set, by theorem 3.17 $f([a,b])$ is also connected subset of $\mathbb{R}$. This implies
$f([a,b])$ is an interval by theorem 2.16,
since $f(a)$ and $f(b)$ is in $f([a,b])$, $\exists x \in [a,b]$ s.t $f(x) = c$
($\because f(a) < c < f(b)$)

Recall that the definition of continuity. For a given point p and a function f continuous at p, for a given $\varepsilon > 0$, $\exists \delta > 0$ which depends on the point a.

$f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$ $\rightarrow$ we find $\cancel{(1+|a|)} \dfrac{\varepsilon}{\cancel{(1+|a|)}}$
and solve

If for some continuous function we can choose the same δ for all p then we say that the function is uniformly cont.

Defn: Suppose $(X,d), (Y,d')$ are metric spaces and $f: X \to Y$ is a function, then f is said to be uniformly continuous on a subset A of X if
$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t $\forall x, p \in A$
$$d(x,p) < \delta \Rightarrow d'(f(x), f(p)) < \varepsilon$$
(if $d(x,p) < \delta$ then $d'(f(x), f(p)) < \varepsilon$,
so note that the δ remains same)

Example: 1) Let $f: (0,1) \to \mathbb{R}$ defined as $f(x) = \dfrac{1}{x}$.
The function f is continuous as x is cont, $\dfrac{1}{x}$ is cont for $x \in (0,1]$
$\rightarrow$ as $x \in (0,1)$

Claim: f is not uniformly continuous
Let $\varepsilon = 5$ then Assume $0 < \delta < 1$. Choose $p = \delta$ and
$x = \dfrac{\delta}{6}$, then $|p - x| = \left|\dfrac{5\delta}{6}\right| < \delta$, but $|f(p) - f(x)| = \left|\dfrac{1}{p} - \dfrac{1}{x}\right|$
$$= \left|\frac{1}{\delta} - \frac{6}{\delta}\right|$$

$\left|\frac{\delta}{\delta}\right| > 5 \quad \therefore$ f is not uniformly continious

2) let $f : [0,1] \to \mathbb{R}$ defined as $f(x) = x^2$. This is uniformly continious which can be seen as follows.

let $\varepsilon > 0$, $\delta = \varepsilon/2$ and $|x - p| < \delta$

$$|f(x) - f(p)| = |x^2 - p^2| = |x-p||x+p| \leq (|x|+|p|)|x-p|$$
$$\leq 2\,|x-p|$$
$$< \cancel{2} \times \varepsilon/\cancel{2} = \varepsilon$$

$\therefore$ So, f is uniformly continious.

Theorm 3.19 : let $f : X \to Y$ be a function where (X, d), (Y, d') are metric spaces. If X is compact and f is continous, then f is uniformly cont.

23rd sept:

Theorem 3.19: Let $f: X \to Y$ be a function where (X,d), (Y,d') are metric spaces. If X is compact and f is continuous, then f is uniformly cont.

(continuity: $f: X \to Y$, at a point $a \in X$, if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t $d(x,a) < \delta \Rightarrow d'(f(x), f(a)) < \varepsilon$

uniform continuity: f is uniformly cont, for $\varepsilon > 0$, $\exists \delta > 0 \to \delta$ works for all points)

($f(x) = x \leftarrow$ uniform cont
$f(x) = x^2$ cont, on $[0,1] \leftarrow X$ is compact $\Rightarrow f(x) = x^2$ is uniformly cont)

proof: (Idea: start $\varepsilon > 0$, find δ)

Let $\varepsilon > 0$, be given. Since f is cont for each point $a \in X$, $\exists r > 0$ (depending on a) s.t

$$d'(f(x), f(a)) < \varepsilon/2 \text{ for } x \in B(a, r)$$

the collection

$$\{B(a, r/2) \mid a \in X\} \text{ is a covering for}$$

compact metric space X. So $\exists a_1, a_2 \ldots a_n \in X$

$$X \subset \bigcup_{i=1}^{n} B(a_i, r_i/2)$$

$$\delta = \min\{r_i/2 \mid i = 1, 2, \ldots n\}$$

Claim: This δ works for uniform continuity for the ε.
Let $x, p \in X$, s.t $d(x,p) < \delta$

Because $\{B(a_i, \frac{r_i}{2}) \mid i = 1, 2, \ldots n\}$ is an open cover, $\exists k$ s.t $1 \leq k \leq n$ and $x \in B(a_k, \frac{r_k}{2})$

Therefore $d'(f(x), f(a_k)) < \varepsilon/2$
by triangle inequality

$$d(p, a_k) \leq d(p, x) + d(x, a_k)$$
$$< \delta + \frac{r_k}{2} \leq \frac{r_k}{2} + \frac{r_k}{2} = r_k$$

$\therefore p \in B(a_k, r_k)$
so $d'(f(p), f(a_k)) < \varepsilon/2$

$$\text{so } d'(f(p), f(x)) \leq d'(f(p), f(a_k)) + d'(f(a_k), f(x))$$
$$< \varepsilon/2 + \varepsilon/2$$
$$= \varepsilon$$

$$d'(f(p), f(x)) < \varepsilon$$

Def 4: $f: (a,b) \to X$, for some metric space X, then any point $x \in [a,b)$ if $f(t_n) \to q$ as $n \to \infty$ for all seq in $\{t_n\}_{n=1}^{\infty}$ in (x, b) st $t_n \to x$, then we say q is the right hand limit of f and

$$\lim_{t \to x^+} f(t) = q \quad \text{or } f(x^+) = q$$

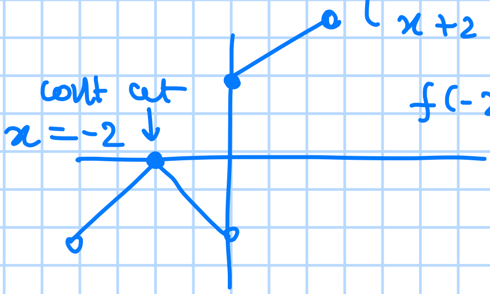
sim, $\lim_{t \to x^-} f(t)$ or $f(x^-)$

Note: we say x is discontinuity of f if f is not cont. at x. In this case one of the following conditions are satisfied.

(a) Either $f(x^+)$ or $f(x^-)$ does not exist.
(b) Both $f(x^+), f(x^-)$ exist but have diff values
(c) Both $f(x^+), f(x^-)$ exist and $f(x^+) = f(x^-) \neq f(x)$

The discontinuity is said to be removable if it is of type (c). Others are non-removable

Eg: (I) Define $f(x) = \begin{cases} x+2 & -3 \leq x < -2 \\ -x-2 & -2 \leq x < 0 \\ x+2 & 0 \leq x < 1 \end{cases}$



$$f(-2^-) = f(-2^+) = f(-2) = 0$$

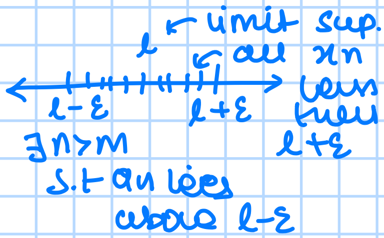
(II) $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right); & x \neq 0 \\ 0 \; ; & x = 0 \end{cases}$

f is cont at all points except $x=0$
$f(0^+)$ and $f(0^-)$ do not exist

This falls in category (a) discontinuity

Defn: Let $\{a_n\}_{n=1}^{\infty}$ be a seq of real numbers.

Suppose $\exists\, l \in \mathbb{R}$ satisfying the following:



i) For every $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t $a_n < l + \varepsilon \;\; \forall\, n \geq n_0$

ii) Given $\varepsilon > 0$ and $m \in \mathbb{N}$, $\exists\, n > m$ s.t $a_n > l - \varepsilon$

Then l is called limit supererior of $\{a_n\}_{n=1}^{\infty}$

and we write $l = \limsup_{n\to\infty} a_n$

If seq is not bouded above, then we define $\limsup_{n\to\infty} a_n = \infty$

If seq is bounded above but not bouded below, and no finite limsup, then we say

$\limsup_{n\to\infty} a_n = -\infty$ ($\{-n\}_{n=1}^{\infty}$ is an example)

The limit inferior of $\{a_n\}_{n=1}^{\infty}$ is defined as:

$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty} b_n$ where $b_n = -a_n \;\; \forall\, n \in \mathbb{N}$

(Traditional, $\lim_{n\to\infty} \sup\{a_n, a_{n+1} \ldots \} = \limsup_{n\to\infty} a_n$)

Example: 1) $a_n = (-1)^n$, $\liminf_{n\to\infty} a_n = -1$, $\limsup_{n\to\infty} a_n = 1$ → prove

2) $a_n = (-1)^n\left(1 + \frac{1}{n}\right)$, $\liminf_{n\to\infty} a_n = -1$, $\limsup_{n\to\infty} a_n = 1$ → prove

3) If $\{a_n\}_{n=1}^{\infty}$ conv to $l \in \mathbb{R}$, then
$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = l$ → prove

Note : (i) $\underline{\lim}\, a_n \leq \overline{\lim}\, a_n$

(ii) If $\exists n_0 \in \mathbb{N}$ s.t $a_n \leq b_n$, $\forall n \geq n_0$
then $\underline{\lim}\, a_n \leq \underline{\lim}\, a_n$

$\overline{\lim}\, a_n \leq \overline{\lim}\, b_n$

eg: $a_n = (-1)^n$

now for $\overline{\lim}\, a_n$, say $l = 1$, then $\forall \varepsilon > 0$,

as $\varepsilon > 0$
$\Rightarrow 0 < \varepsilon$
$\Rightarrow 1 + 0 < 1 + \varepsilon$
$\Rightarrow (-1)^n < 1 + \varepsilon$ $\left(\because (-1)^n = 1 \text{ or } -1\right)$
so $\forall n \geq 1$, $(-1)^n < 1 + \varepsilon$

and for $\varepsilon = 0.5$ $\quad 1 - 0.5 = 0.5$
$(-1)^{2n} > 0.5$ $\quad$ (Example ε, for $\forall \varepsilon$ true)

$\therefore \overline{\lim}\, a_n = 1$
similarly $\underline{\lim}\, a_n = -\overline{\lim}\, b_n = -1$

eg: $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$

Here $\overline{\lim}\, a_n$ say 1
then
$\forall \varepsilon > 0$,
using archimedean property, $\exists n_0 \in \mathbb{N}$
s.t $\frac{1}{n_0} < \varepsilon$

$\Rightarrow 1 + \frac{1}{n_0} < 1 + \varepsilon$

$\Rightarrow (-1)^n \left(1 + \frac{1}{n_0}\right) < 1 + \varepsilon$

$\Rightarrow (-1)^n \left(1 + \frac{1}{n}\right) < 1 + \varepsilon \quad \forall n \geq n_0$

also for $\varepsilon = 1$
$1 - 1 = 0$
$(-1)^n \left(1 + \frac{1}{n}\right) > 0$ $\quad$ ($\forall \varepsilon$ also true, this is an example)
$\forall$ odd n

$\therefore \overline{\lim}\, a_n = 1$
similarly $\underline{\lim}\, a_n = -\overline{\lim}\, b_n = -1$

eg: $\{a_n\}_{n=1}^{\infty} \to l$
then $\overline{\lim}\, a_n = \underline{\lim}\, a_n = l$

Now, as $a_n \to l$ $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t
$|a_n - l| < \varepsilon \quad \forall n \geq n_0$
now this means that
$a_n - l < \varepsilon$

now $a_n - l > -\varepsilon$
$\Rightarrow a_n > l - \varepsilon \quad \forall \varepsilon > 0$, possible

similarly for $\underline{\lim}\, a_n = -\overline{\lim}\, (-a_n)$

24th sept :

3. Series :

Defn: A series $\sum_{n=1}^{\infty} a_n$, where $a_n \in \mathbb{R}$ or $\mathbb{C}$ denotes a sequence $\{s_n\}_{n=1}^{\infty}$ where

$$s_n = a_1 + \cdots a_n = \sum_{k=1}^{n} a_k \; , \; \{s_n\}_{n=1}^{\infty} = \left\{ \sum_{k=1}^{n} a_k \right\}_{n=1}^{\infty}$$

s_n = partial sums of $\sum_{n=1}^{\infty} a_n$

A series is said to be converge if $\{s_n\}_{n=1}^{\infty}$ converges.

Example: The partial sum of $\sum_{i=0}^{n} x^i = 1 + x + x^2 + \cdots x^n$ is $\frac{(1-x^{n+1})}{(1-x)}$ (for $x \neq 1$)

if $0 < x < 1$ and $n \to \infty$ then

$$\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$, for $x \geq 1$ series diverges

Theorem 3.1: (Cauchy condition for series)
The series $\sum_{n=1}^{\infty} a_n$ converges iff $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t $\forall n \geq n_0$

$$|a_{n+1} + \cdots\cdots a_{n+p}| < \varepsilon \quad \forall p \in \mathbb{N}$$

proof: let $s_n = \sum_{k=1}^{n} a_k$ then

$|s_{n+p} - s_n| < \varepsilon \quad \forall p \in \mathbb{N}$ is written as above

so we can restate theorem as

$\{s_n\}_{n=1}^{\infty}$ converges iff $\{s_n\}_{n=1}^{\infty}$ is cauchy in $\mathbb{R}$

as $\mathbb{R}$ is complete this is true. so
$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t $\forall n \geq n_0$

$\{s_n\}_{n=1}^{\infty}$ converges if $|a_{n+1} + a_{n+2} + \cdots a_{n+p}| < \varepsilon \; \forall p \in \mathbb{N}$

Remark: Taking $p = 1$, it follows that $\lim_{n \to \infty} a_n = 0$

Note: If $\{s_n\}_{n=1}^{\infty}$ is convergent, then $\lim_{n\to\infty} a_n = 0$, but $\lim_{n\to\infty} a_n = 0$ does not imply $\sum_{n=1}^{\infty} a_n$ is convergent.

disproof example: $a_n = \frac{1}{n}$, as $n \to \infty$
$a_n = \frac{1}{n} \to 0$

let $\varepsilon = 1/2$ for any $n_0 \in \mathbb{N}$
then choose $m \in \mathbb{N}$
s.t $2^m \geq n_0$

for $n = 2^m$
$p = 2^m$ we get

$$a_{n+1} + \cdots a_{n+p} = \frac{1}{2^m + 1} + \cdots \frac{1}{2^m + 2^m} \geq \frac{2^m}{2^m + 2^m} = \frac{1}{2}$$

$\therefore$ cauchy condition not satisfied
$\Rightarrow \sum \frac{1}{n}$ not conv

Theorem 3.2 : Suppose
(a) The partial sum A_n of $\sum a_n$ form bounded sequence
(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$
(c) $\lim_{n\to\infty} b_n = 0$

then $\sum a_n b_n$ converges

proof: since $\{A_n\}_{n=1}^{\infty}$ is bounded, $\exists M>0$ s.t

$|A_n| \leq M, \ \forall n \in \mathbb{N}$

let $\varepsilon > 0$

since $\lim_{n\to\infty} b_n = 0$

$\exists n_0 \in \mathbb{N}$ s.t

$b_{n_0} \leq \varepsilon / 2M$

also for $q > p \geq n_0$

$$\left|\sum_{n=p}^{q} a_n b_n\right| = \left|\sum_{n=p}^{q} (A_n - A_{n-1}) b_n\right|$$

$$= \left|\sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n\right|$$

$$= \left| A_p b_p + A_{p+1} b_{p+1} + \cdots \cdots A_q b_q - (A_{p-1} b_p + A_p b_{p+1} \cdots A_{q-1} b_q)\right|$$

$$= \left|\sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right|$$

$$\leq M\left(\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p\right)$$

as $b_n \geq b_{n+1}$

$\sum b_n - b_{n+1}$

$= b_p - b_q$

$$\leq M(b_p - \cancel{b_q} + \cancel{b_q} + b_p)$$

$$\leq 2M b_{n_0} \leq \varepsilon$$

so $\sum a_n b_n$ converges by Cauchy condition

Defn: If $a_n > 0, \forall n \in \mathbb{N}$, the series $\sum (-1)^{n+1} a_n$ is called alternating series

Theorm 3.3: (Leibnitz test)

If $\{b_n\}_{n=1}^{\infty}$ is decreasing seq. converging to 0. the alternative series $\sum (-1)^{n+1} b_n$ converges.

proof: as $\{b_n\}_{n=1}^{\infty}$ is decreasing $b_1 \geq b_2 \geq \cdots$ and $\lim_{n\to\infty} b_n = 0$

the series $s_n = \sum (-1)^{n+1}$ is bounded

we have $\sum (-1)^{n+1} b_n$ conver

follows from theorm 3.2

26th sept :

Defn : $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converges.

Theorm 3.4 : Absolute convergence of $\sum_{n=1}^{\infty} a_n$ implies convergence.

proof : we have the inequality:

$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| \leq |a_{n+1}| + \cdots |a_{n+p}|$

using the inequality in cauchy condition of $\sum |a_n|$ we get absolute convergence of $\sum a_n$ implies conv.

(Converse of Theorm 3.4 is not true as alternating series $\sum (-1)^{n+1} \frac{1}{n}$ converges, but $1/n$ does not converge)

Theorm 3.5 : (comparision test)

If $a_n \geq 0$ and $b_n \geq 0$, $\forall n \in \mathbb{N}$ and if $\exists c > 0$ and $n_0 \in \mathbb{N}$ s.t $a_n < c b_n$ $\forall n \geq n_0$, then convergence of $\sum b_n$ implies convergence of $\sum a_n$.

proof : let $\sum b_n$ be convergent. so the partial sum of $\sum b_n$ are bounded above. Thus partial sum of $\sum a_n$ are bounded above and increasing. Hence $\sum a_n$ is conv.

as $a_n < c b_n$
$\forall n \geq n_0$

$\Rightarrow |a_{n+1} + a_{n+2} + \cdots a_{n+p}|$
$< c|b_{n+1} + b_{n+2} + \cdots b_{n+p}|$
$< c\varepsilon$

or, $\sum a_n < c(M)$

$\downarrow$ as $\sum b_n$ conv

$\sum a_n$ is bounded above and as $a_n \geq 0$ $\forall n$, $\sum a_n$ is increasing

$\therefore \sum a_n$ is convergent.

Theorm 3.6 : (limit comparision test)

Assume $a_n > 0$, $b_n > 0$, $\forall n \in \mathbb{N}$ and suppose

$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, then

the $\sum a_n$ conv $\Leftrightarrow$ $\sum b_n$ conv

proof : let $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\exists n_0 \in \mathbb{N}$

s.t $c/2 < \frac{a_n}{b_n} < 3c/2$ $\forall n \geq n_0$

Applying theorm 3.5 it is clear that

$a_n < 3c/2\, b_n$ and $c/2\, b_n < a_n$

so, $\sum a_n$ conv $\Rightarrow$ $\sum b_n$ conv and

$\sum b_n$ conv $\Rightarrow$ $\sum a_n$ conv

(Note : If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $\frac{a_n}{b_n} < \varepsilon$ for some $\varepsilon > 0$

$\therefore a_n < \varepsilon b_n$ or
if $\sum b_n$ conv $\Rightarrow$ $\sum a_n$ conv)

Theorem 3.7: (Ratio test)

Given a series $\sum a_n$ of non-zero terms, let $r = \underline{\lim} \left|\frac{a_{n+1}}{a_n}\right|$, $R = \overline{\lim} \left|\frac{a_{n+1}}{a_n}\right|$

a) The series $\sum a_n$ conv absolutely if $R<1$

b) The series $\sum a_n$ diverges if $r>1$

c) The test is inconclusive if $r \leq 1 \leq R$

proof: (a) let $R<1$. choose x s.t $R<x<1$

By defn of $\overline{\lim}$ $\exists N$ s.t $\left|\frac{a_{n+1}}{a_n}\right| < x$ $\forall n \geq N$

($\overline{\lim} \left|\frac{a_{n+1}}{a_n}\right| = R$, then $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t $\left|\frac{a_{n+1}}{a_n}\right| < R+\varepsilon$ $\forall n \geq n_0$)

so $\left|\frac{a_{n+1}}{a_n}\right| < \frac{x^{n+1}}{x^n}$

$\Rightarrow \frac{|a_{n+1}|}{x^{n+1}} < \frac{|a_n|}{x^n} \leq \frac{|a_N|}{x^N}$ $\forall n \geq N$

$\Rightarrow |a_n| \leq \frac{|a_N|}{x^N} x^n$ $\forall n \geq N$

$\Rightarrow |a_n| \leq C x^n$ $\forall n \geq N$ where $C = |a_N| x^{-N}$

or $|a_n| < C x^n$

as $R<x<1$

$\sum x^n$ conv, $\therefore$ $|a_n|$ conv by comparison test

$\Rightarrow$ a_n conv by comp test

(b) let $r>1$, so by defn of $\underline{\lim}$ $\exists N$ s.t

$\frac{|a_{n+1}|}{|a_n|} > 1$ $\forall n \geq N$

($\underline{\lim} \left|\frac{a_{n+1}}{a_n}\right| = r$, then $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t

$= -\overline{\lim} - \left|\frac{a_{n+1}}{a_n}\right|$ $\quad -\left|\frac{a_{n+1}}{a_n}\right| < -r+\varepsilon$ $\forall n \geq n_0$

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| > r-\varepsilon$ $\forall n \geq n_0$)

Now $\left|\frac{a_{n+1}}{a_n}\right| > 1$ $\forall n \geq N$

$\Rightarrow |a_{n+1}| > |a_n|$ $\forall n \geq N$

$\Rightarrow \lim_{n\to\infty} a_n \neq 0$ $\therefore$ $\sum a_n$ diverges

(c) consider $\sum 1/n$ and $\sum 1/n^2$. In both the series $r=R=1$ ($\lim_{n\to\infty} \frac{n}{n+1} = 1$, $\lim_{n\to\infty} \left(\frac{n^2}{(n+1)^2}\right) = 1$)

but still $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n^2}$ conver

Theorm 3.8: (Root test)
let $\sum a_n$ be a series and let $\rho := \overline{\lim} \sqrt[n]{|a_n|}$, then the following holds:

a) $\sum a_n$ conv absolutely if $\rho < 1$

b) $\sum a_n$ diverges if $\rho > 1$

c) The test is inconclusive if $\rho = 1$

proof:
(a) let $\rho < 1$, $\exists x$ s.t $\rho < x < 1$
by defn of $\overline{\lim}$, $\exists N \in \mathbb{N}$ s.t

$$\sqrt[n]{|a_n|} < x \quad \forall n \geq N$$

$$|a_n| < x^n \quad \forall n \geq N$$

as x^n converges

$\Rightarrow |a_n|$ converges (∵ comparison test)
$\Rightarrow a_n$ conv

(b) let $\rho > 1$, so by defn of $\overline{\lim}$, $|a_n| > 1$ infinitely often

$\therefore \lim\limits_{n\to\infty} a_n \neq 0$

so $\sum a_n$ diverges

(c) $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges but $\rho = 1$ for both the series (∵ $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, $\lim\limits_{n\to\infty} (n)^{2/n} = 1$)

Theorm 3.9: (Cauchy condensation test)
suppose $\{a_n\}_{n=1}^{\infty}$ is a decreasing seq of positive real numbers. Then $\sum\limits_{1}^{\infty} a_n$ conv iff

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + \cdots \text{ conv}$$

30^{th} sep:

Recap: started with series
① comparison test ($a_n < c b_n$)
② limit comp test (b_n/a_n)

③ Ratio test, root test, leibnitz test

Theorm 3.9: (Cauchy condensation test)
suppose $\{a_n\}_{n=1}^{\infty}$ is a decreasing seq of positive real numbers. Then $\sum_1^{\infty} a_n$ cong iff

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + \cdots \text{ cong}$$

$$\left(\text{series}_1 = \sum_{1}^{\infty} a_n \text{ cong iff series}_2 \sum_{n=0}^{\infty} 2^n a_{2^n} \right)$$
$n \to \infty$ decreasing terms, $n \to \infty$

Example: $\text{series}_1 = \left(\frac{1}{1^3}+\frac{1}{1}\right)+\left(\frac{1}{2^3}+\frac{1}{2}\right)+\left(\frac{1}{3^3}+\frac{1}{3}\right)+\left(\frac{1}{4^3}+\frac{1}{4}\right)+\cdots$

Note this does not converge as $1+\frac{1}{2}+\cdots\frac{1}{n}+\cdots$ does not converge.
but see $1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots$ converges

Example: $(-1)^n \frac{1}{n}$ is cong series, but if n is even then diverge

proof: let S_n and t_k be the partial sums of the series $\sum a_n$ and $\sum 2^k a_{2^k}$
for $n < 2^k$

$$S_n \leq \underset{1}{a_1} + \underset{2}{(a_2+a_3)} + \cdots + \underset{2^k}{(a_{2^k}+\cdots+a_{2^{k+1}-1})}$$
every bracket has 2^k terms

$$S_n \leq a_1 + 2a_2 + 4a_4 + \cdots 2^k a_{2^k}$$
as $a_{n_0} \geq a_{n_0+1}$ (decreasing seq) $\forall n_0 \in \mathbb{N}$

$$\Rightarrow \boxed{S_n \leq t_n} \text{ for } n < 2^k$$

for $n > 2^k$

$$S_n \geq a_1 + a_2 + (a_3+a_4) + \cdots + (a_{2^{k-1}}+\cdots a_{2^k})$$
$$\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \cdots 2^{k-1} a_{2^k}$$
$$S_n = \frac{1}{2} t_k$$
$$\boxed{2S_n \geq t_k} \text{ for } n > 2^k$$

Thus $\{S_n\}_{n=1}^{\infty}$ and $\{t_k\}_{k=1}^{\infty}$ are either both bounded or both unbounded.
since both are inc sequences
$\{S_n\}_{n=1}^{\infty}$ is cong iff $\{S_n\}_{n=1}^{\infty}$ is bounded
iff $\{t_n\}_{n=0}^{\infty}$ is bounded iff $\{t_n\}_{n=0}^{\infty}$ is cong.
$\therefore \{S_n\}_{n=1}^{\infty}$ is cong $\Leftrightarrow$ $\{t_n\}_{n=0}^{\infty}$ is cong

Theorem 3.10 : $\sum \frac{1}{n^p}$ converges if $p>1$, diverges if $p \le 1$.

proof : if $p \le 0$, then $\sum \frac{1}{n^p}$ diverges as $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$

Assuming for $b>1$, $p \in \mathbb{R}$, b^p is well-defined, also x^p for $p>0$

b^p is increasing as $b>1$, $p>0$

as $\frac{1}{n^p} = (n)^{-p}$

$-p \geq 0$

or $(n)^{-p}$ is increasing

so, $\sum \frac{1}{n^p}$ diverges $\forall$ $p \le 0$.

for $p>0$,

$\sum \frac{1}{n^p}$ conv iff $\sum 2^k \frac{1}{(2^k)^p}$ conv

$\sum 2^{k-kp}$ conv

this is a geometric series with ratio 2^{1-p}

if $2^{1-p}<1$ conv

$2^{1-p} \geq 1$ divg $\Rightarrow 2^{1-p} \geq 2^0 \Rightarrow 1-p \geq 0 \Rightarrow \boxed{p \le 1}$ divg

$2^{1-p} < 2^0$

$1-p<0$

or $\boxed{p>1}$ conv

Theorem 3.11: (Telescopic series)

Let $\{a_n\}_1^\infty, \{b_n\}_1^\infty$ be two seq s.t $a_n = b_{n+1} - b_n$ for $n=1, \ldots$

then '$\sum a_n$' conv iff $\lim_{n\to\infty} b_n$ exist

in this case $\sum_1^\infty a_n = \lim_{n\to\infty} b_n - b_1$

$$\left(\begin{aligned} a_k &= b_{k+1} - \cancel{b_k} \\ a_{k-1} &= \cancel{b_k} - \cancel{b_{k-1}} \\ &\vdots \\ a_1 &= \cancel{b_2} - b_1 \end{aligned} \Rightarrow \sum a_k = b_{k+1} - b_1 \right)$$

proof : $\sum_{k=1}^n a_k = b_{n+1} - b_1$

so, $\sum_{k=1}^\infty a_k = \lim_{n\to\infty} b_n - b_1$

The exponential : The series $\sum_{n=0}^\infty \frac{1}{n!}$ is conv because the seq of partial sums is increasing

and $s_n = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots \frac{1}{1\cdot 2\cdot 3 \cdots n}$

$< 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 2} + \frac{1}{2\cdot 2\cdot 2} + \cdots \frac{1}{2^{n-1}}$

$< 1 + \frac{1}{1-\frac{1}{2}} = 1+2 = 3$

$s_n < 3$

$\therefore$ s_n are bounded by 3 and increasing seq

$\therefore$ s_n converges.

Defn : $e = \sum_{n=0}^\infty \frac{1}{n!}$

The log: fix $b>1$, $y>0$. It can be shown a unique $x \in \mathbb{R}$ s.t $b^x = y$. we define 'logarithm of y wrt base b' as

$$x = \log_b y$$ (see exercise 7 chapter 1, Rudin)

we write $\log s$ for $\log_e s$

Defn: (logarithm function)

$$y = f(x) = \log_b x \leftarrow \text{base } b$$

logarithm of x wrt b

Remark: log is a monotonically increasing function

Theorm 3.12: If $p>1$, the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges. If $p \le 1$, the series diverges

proof: The seq $\left\{ \frac{1}{n(\log n)} \right\}_{n=2}^{\infty}$ is decreasing.

By cauchy condensation test,

$$\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=1}^{\infty} \cancel{2^k} \frac{1}{\cancel{2^k}(\log 2^k)^p}$$

$$= \sum_{k=1}^{\infty} \frac{1}{(k \log 2)^p}$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{k^p \log 2} \right)$$

now, $\sum_{n=2}^{\infty} \left(\frac{1}{n(\log n)^p} \right)$ conv $\Leftrightarrow \sum_{k=1}^{\infty} \frac{1}{k^p \log 2}$ conv

$\Leftrightarrow$ $p>1$ then conv, $p \le 1$ then div

so, $\sum_{n=2}^{\infty} \left(\frac{1}{n(\log n)^p} \right)$ conv for $p>1$, div for $p \le 1$.

Rearrangement of Series:

$$\sum a_n = \underbrace{1 - \tfrac{1}{2}}_{1/2} + \underbrace{\tfrac{1}{3} - \tfrac{1}{4}}_{1/12} + \dots \dots > \tfrac{1}{2}$$

now if we rearrange it to $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots$

$$= \left(\tfrac{1}{2} - \tfrac{1}{4}\right) + \left(\tfrac{1}{6} - \tfrac{1}{8}\right) + \left(\tfrac{1}{10} - \tfrac{1}{12}\right) + \dots$$

$$= \tfrac{1}{2}\left(1 - \tfrac{1}{2}\right) + \tfrac{1}{2}\left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \tfrac{1}{2}\left(\tfrac{1}{5} - \tfrac{1}{6}\right) \dots$$

$$\sum a_n = \tfrac{1}{2}\Big(\underbrace{1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots}_{\sum a_n} \Big)$$

Theorm 3.13 : (Riemann)

A series which is not absolutly conv but conv, then for $x, y \in (-\infty, \infty)$ s.t $x \leq y$.

Then $\exists$ a rearragement Σb_n of Σa_n s.t for $t_n = b_1 + b_2 + \cdots b_n$

$\underline{\lim}\, t_n = x \qquad \overline{\lim}\, t_n = y$

Proof is too lenthy, so it will be skipped.