# MA401
# 114

Ref: (I) linear algebra: Hoffman & Kunze
(II) linear algebra: S. Lang
(III) linear algebra: P. Lax

Eval: 2 quizes (sup)
midsem
Endsem

29th July: $\Gamma$ vector spaces

$M_{n\times n}(\mathbb{C})$ - Algebra

Examples where lin alg. pop up: ① Markov's matrix ② adjacent matrix ③ $GL_{n\times n}(\mathbb{C})$ group
↳ see my notes

*** gaussian method:**

$Ax = b$ — we have to solve for $x$.

$x_1 + 3x_2 + x_3 = 2$
$x_1 + 8x_2 + x_3 = 12$
$4x_2 + x_3 = 2$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 1 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

3 planes → intersection point ($\mathbb{R}^3$)

coeff. matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}_{3\times3} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1} \quad b = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}_{3\times1}$$

$$Ax = \left\{ x_1\underbrace{c_1 + x_2c_2 + x_3c_3}_{\text{column vector}} ; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \right\}$$

colum space (thisset)

if $b \in Ax$ then there is a soln.

**Elementary Row operations:**

1. multiply non-zero scalar to row.
2. multiply non-zero scalar to a row and add/ subst. another row.
3. Interchange rows.

pivot

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{R_2' = R_2 - R_1} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 5 & 0 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{R_3' = R_3 - \frac{4}{5}R_2} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 1 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 5 & 0 & 10 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 5 & 0 & 10 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

$ux = b'$ $\quad x_3 = -6$
$5x_2 = 10 \Rightarrow x_2 = 2$
$x_1 + \cancel{6} - \cancel{6} = 2$
$\Rightarrow x_1 = 2$

note: $$[x_1 \ x_2 \ x_3]_{1\times3} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}_{3\times3} = x_1R_1 + x_2R_2 + x_3R_3$$

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

here, $$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & 5 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

by $$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ by:}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4/5 & 1 \end{bmatrix} \text{ and then } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

see $E_{ij}$ matrices

*** Invertable: $AB = BA = I$ (Def$^n$)** → see

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

see perm. matrix ←
$$\underset{P_{12}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{P_{12}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = I$$

if $Ax = b$
$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & c_1 \\ 0 & 5 & 1 & c_2 \\ 0 & 0 & 0 & c_3 \end{array}\right]$$
not solvable as $x_3 . 0 = c_3$
$x_3$ can be anything

now $Ax = b$
$$\underbrace{\cdots E_2 E_1}_{} A = u$$
$EA = u$
so $\underset{u}{EA}x = ux = Eb = b'$

Note: $Ax = b$
$ux = b'$
$\Leftrightarrow EAx = Eb$
$\Leftrightarrow Ax = b$
so, $ux = b'$
$\Leftrightarrow Ax = b$

$1^{st}$ Aug : $A \in M_{n\times n}(\mathbb{C})$

$EA = U \leftarrow$ upper triangular matrix

↳ we left multiply by

$E_{ij}(\lambda)$ matrices or $P_{ij}$ matrices. } → see notes (all linear algebra MA110 till midsem)

$\begin{bmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2' = R_2 - \alpha R_1$ $E_{21}(-\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = E^{-1}U$

$LU$ where $E^{-1} = L$

as $(E_{21}(\lambda))^{-1} = (\underbrace{E_{21}(-\lambda)}_{\text{Always } L})$

**Remark**: If there is no row exchange in elimination process for $A$ then we can write $A = LU$

**Now**: $A \in M_{m\times n}(\mathbb{C})$, $m < n$

$m\begin{bmatrix} \overset{n}{\ } \end{bmatrix}$

Eg: $m=3$ $n=4$

$U = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $Ux = 0$ then

2 pivots

also $Ax = 0$ (Homogenious equation)

$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$

$x = \begin{pmatrix} -2x_2 - 2x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -3 \\ 1 \end{pmatrix}$

**Note**: In such a case we will always get a non-trivial solution.

(max 3 pivots, 4 variables)

$x_1 = -2x_2 - 2x_4$
$x_3 = 0x_2 - 3x_4$ } $x_1, x_3$ are line comb of $x_2, x_4$ → soln is linear comb of $x_2$ and $x_4$

**Theorm**: Let $A$ be a $n\times n$ matrix. Then $A$ is invertible iff $Ax = 0$ has a unique soln.

**Proof**: ⇐ Let $U$ be the row-reduced Echlon form of $A$. Then the system $UX = 0$, also has a unique solution.

If $r$ is the number of pivots (or number of non-zero rows) in $U$, then

$r = n$

i.e $\underbrace{\text{no. of pivots}}_{\text{no of non-zero rows}} = \text{no. of rows}$

Since every row of $U$ has non-zero element,

$U = I \Rightarrow A = E^{-1}U$

$\Rightarrow A$ is invertibl

(⇒) $A$ is invertible

$Ax = 0$

$\Rightarrow x = A^{-1}\cdot 0$

$\Rightarrow x = 0$ (unique soln)

**Def^n**: A vector space $V$ over a field $F$ is a set equipped with the operations:

(i) $\alpha + \beta \in V$, $\alpha, \beta \in V$

(ii) $c\alpha \in V$, $c \in F$, $\alpha \in V$

s.t

(i) $(V, +)$ is commutative group

(ii) $1.\alpha = \alpha$, $\forall \alpha \in V$

(iii) $c_1(c_2\alpha) = (c_1c_2)\alpha$, $\forall c_1, c_2 \in F$, $\alpha \in V$

(iv) $(c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$, $\forall c_1, c_2 \in F$, $\alpha \in V$

(v) $c(\alpha + \beta) = c\alpha + c\beta$, $\alpha, \beta \in V$, $c \in F$

} See this properties carefully

Exe: $V, W$ are vector spaces over $F$, then

$V \times W = \{(v, w) \mid v \in V, w \in W\}$ is a vector space.

Example of vector spaces :

(i) $\mathbb{R}^2$

(ii) $\mathbb{R}^n$

(iii) $M_{m\times m}(\mathbb{C})$

(iv) $\{f:\Omega\to F\}$ is a vector space over $F$.

$(f+g)(s)=f(s)+g(s)\ \forall s\in\Omega$

$(\alpha f)(s)=\alpha\cdot f(s),\ \forall s\in\Omega,\ \alpha\in F$.

(v) $P_n(x)=\{a_0+a_1x+\cdots a_nx^n \mid a_i\in F\}$ is also a vector space.

(vi) $\{x\in\mathbb{C}^n \mid Ax=0\}=\underset{\text{Null space}}{N(A)}=$ vector space

Non-Example : ① $\mathbb{Q}$ over $\mathbb{R}$

as $c\alpha\notin\mathbb{Q}$ when $c\in\mathbb{R}$ and $\alpha\in\mathbb{Q}$

② $x=2,\ y\in\mathbb{R}$ $(2,y)$ not a vector space.

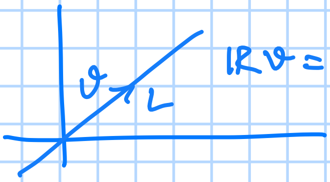
③ $Ax=b$

$b\neq\{0\}$

solution space

(Non Example)

**Def^n** : Let $V$ be a vector space over $F$. A subset $W$ of $V$ is a subspace of $W$ is a vector space with respect to the operations borrowed from $V$.



$\mathbb{R}\vartheta=\{\alpha\vartheta \mid \forall\alpha\in\mathbb{R}\}$

① $c_1\vartheta+c_2\vartheta\in\mathbb{R}\vartheta$

② $0\in\mathbb{R}\vartheta$

③ if $\vartheta\in\mathbb{R}\vartheta$, $-\vartheta\in\mathbb{R}\vartheta$

} check

**Theorm** : Let $W$ be non-empty subset of a vector space $V$ over $F$, then $W$ is a subspace iff

$c\alpha+\beta\in W,\ \forall c\in F$ and $\alpha,\beta\in W$

$(\Rightarrow)$ Trivial (as $W$ is a subspace $c\alpha+\beta\in F$ is true)

$(\Leftarrow)$ since $0=(-\alpha)+(\alpha)$, then $0\in W$ ← 0 ①

for $\alpha\in W$, $-\alpha=(-1)(\alpha)+0\in W$

so $-\alpha\in W$ ← inverse ②

also for $\alpha,\beta\in W$,

for $c=1$

$\alpha+\beta\in W$ ← closure ③

and for $\forall c\in F$, $\beta=0\in W$ ← scalar multiplication ④

$c\alpha\in W$

Thus $W$ is a vector space.

for $v_1,v_2,\dots v_n\in V$ over $F$, we say $\beta\in V$ is a linear combination of $v_1,v_2,\dots v_n$ if $\exists c_1,c_2,\dots c_n\in F$ s.t $\beta=c_1v_1+\cdots c_nv_n=\sum_{i=1}^{n}c_iv_i$

Exe: $V$, $W$ are vector spaces over $F$, then

$V\times W=\{(v,w) \mid v\in V,\ w\in W\}$ is a vector space.

given $V,W$ are vector spaces.

then

$V\times W=\{(v,w) \mid v\in V,\ w\in W\}$

now for $(\alpha_v,\alpha_w)\in V\times W$

and

$(\beta_v,\beta_w)\in V\times W$

$(\alpha_v+\beta_v,\ \alpha_w+\beta_w)\in V\times W$

as Both $\alpha_v+\beta_v\in V$

& $\alpha_w+\beta_w\in W$

also $c(\alpha v, \alpha w) = (c\alpha v, c\alpha w) \in V \times W$

as $c\alpha v \in V$
and $c\alpha w \in W$

8th Aug :

**Defn**: Let $S$ be a set of vectors in $V$. Then the span of $S$ is the intersection of all subspaces of $V$ which contains $S$.

**Theorm**: let $S$ be a set of vectors in $V$, Then $\text{Span}\, S = \left\{ \sum_{i=1}^{n} c_i v_i \;\middle|\; n \in \mathbb{N}, c_i \in F, v_i \in S \right\}$

Exe ↑ proof of $\text{span}\{x_1, x_2, \dots x_n\} = \left\{ \sum_{i=1}^{n} c_i x_i \;\middle|\; n \in \mathbb{N}, c_i \in F \right\}$

**Defn**: A collection of vectors $x_1, x_2, \dots x_n$ is linearly dependent if

$$\sum_{i=1}^{n} \alpha_i x_i = 0$$

for some choices of $\alpha_i$'s s.t not all zero.

**Defn**: The collection is linearly independent if it is not dependent

$$\sum_{i=1}^{n} \alpha_i x_i = 0 \iff \alpha_1 = \alpha_2 = \cdots \alpha_n = 0$$

Examples : (i) $\{x_1 = 0, x_2, \dots x_n\}$ dependent as $\forall a_1 \neq 0$ and $a_i = 0$, $i \geq 2$

$a_1(x_1) + 0 = 0$ as $x_1 = 0$

Note: A set of vectors $S$ is independent if any finite (non-empty) subset of $S$ is independt.

(ii) Two vectors $x_1$ $x_2$ are dependent $\iff$ $x_1 = \lambda x_2$ for some $\lambda \in F$ or $x_2 = \beta x_1$ for some $\beta \in F$

Eg: $F(X) = \{f : \mathbb{R} \to \mathbb{R}\}$

$\{1, a, \dots x^n\}$ Lin Ind.

proof: $(\Rightarrow)$ if $x_1, x_2$ are dependent, then $\exists\, \alpha_1, \alpha_2 \in F$ s.t not both zero.

$$\alpha_1 x_1 + \alpha_2 x_2 = 0$$

if $\alpha_1 \neq 0$ then

$$x_1 = -\frac{\alpha_2}{\alpha_1} x_2$$

if $\alpha_2 \neq 0$ then

$$x_2 = -\frac{\alpha_1}{\alpha_2} x_1$$

$(\Leftarrow)$ if $x_1 - \lambda x_2 = 0$ then $\exists\, a_1, a_2 \in F$ not both zero s.t

$$a_1 x_1 + a_2 x_2 = 0$$

(iii) Any finite superset of a dependent collections of vectors is dependent.

proof: $\{x_1, x_2, \dots x_n\}$ $\sum_{i=1}^{n} \alpha_i x_i =$

$$\{\underbrace{x_1, x_2, \dots x_n}, \underbrace{y_n, y_{n+1}, \dots}\}$$

$$\sum_{i=1}^{n} \alpha_i x_i + \underbrace{\sum 0 \cdot y_{n+1}}_{0} = 0$$

$\therefore$ dependent.

(iv) Subset of a finite collection of independent vectors is independent again.

proof : $\sim$( finite superset of dep $\Rightarrow$ fin sub dep)

$\sim$(fin sub dep) $\Rightarrow$ $\sim$( fin superset dep)

fin set independent $\Rightarrow$ fin subset independent

Def^n : A set of vectors $S$ in a vector space $X$ is a basis if

(i) span $S = X$

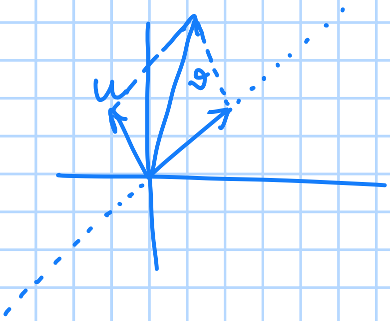
(ii) $S$ is linearly independent

$$\text{span } S = \left\{ \sum_{i=1}^{n} \alpha_i x_i \;\middle|\; x_i \in S,\ \alpha_i \in F \right\}$$

$S = \{x_1, x_2, \ldots x_n\}$

$$x = \sum_{i=1}^{n} \alpha_i x_i = \sum_{i=1}^{n} \beta_i x_i$$

$\Rightarrow \alpha_i = \beta_i$ (as lin independent)




we can choose any other u then $\lambda v$ then $\{u, v\}$ is a basis

① span $\{u, v\} = \mathbb{R}^2$

② $\{u, v\}$ are lin ind.

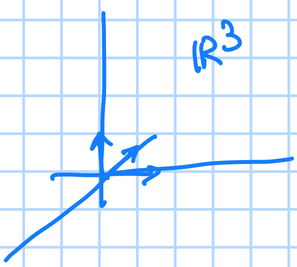
as $\alpha v + \beta u$ will give any vector in $\mathbb{R}^2$.

$\{(1,0), (0,1)\}$ is a basis of $\mathbb{R}^2$

$\{u, v, w\}$ is not a basis of $\mathbb{R}^2$: as $w$ can expressed as $\alpha u + \beta v$

$w - \alpha u - \beta v = 0$

$\therefore$ dependent




$\{\underbrace{(x_1, x_2)}_{\text{plane}}, (x_3)\}$

if $x_3$ is a line on plane then span $\neq \mathbb{R}^3$

lemma : If span $\{x_1, \ldots x_n\} = X$ and if $\{y_1, y_2 \ldots y_m\}$ is linearly independent in $X$ then $m \leq n$.

proof : $y_1 = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha_i \in F$

one of the $\alpha_i \neq 0$ as $y_1 \neq 0$.

without any loss of generality let $\alpha_1 \neq 0$

$$x_1 = \frac{y_1}{\alpha_1} - \sum_{i=2}^{n} \alpha_i x_i$$

claim $\{y_1, x_2, \ldots x_n\} = X$

any $x \in X$

$x = \sum_{i=1}^{n} \beta_i x_i$ $\qquad$ $x_1 = \frac{1}{\alpha_1} y_1 - \sum \alpha_i x_i$

so $x$ can be expressed as l.c of $\{y_1, x_2, x_3, \ldots x_n\}$

repeat again to get:

$\text{span}\{y_1, y_2, x_3, \ldots x_n\} = X$

again to get:

$\text{span}\{y_1, y_2, y_3, \ldots\ldots y_n\} = X$

$y_2 = \gamma_1 y_1 + \underbrace{\sum_{i=2}^{n} \gamma_i x_i}$

some of $\gamma_i \neq 0$

as if all zero

$y_2 = \gamma_1 y_1$ (not possible)

now if $m > n$,

then some $n$ in $\{y_1, y_2, \ldots y_m\}$

let it be $\{y_1, y_2, \ldots y_n\}$ also lin ind.

as $\text{span}\{y_1, y_2, \ldots y_n\} = X$

One more element here should still make the span lin ind.

But it would be lin dep. $\therefore$

$m \leq n$

12th Aug:

**Lemma**: Let $\{a_1, \dots a_n\}$ be a spanning set of vector space X. Then any l.i set of vectors $\leq$ to n-elements.

**proof**: Let $\{x_1, x_2, \dots x_m\}$ be lin in in X, suppose $m > n$.

as $x_i \in X \ \forall i = 1, 2, \dots m$
and $\{a_1, a_2, \dots a_n\}$ span X

$$x_i = \sum_{j=1}^{n} \alpha_{ij} a_j \ , \forall i = 1, 2, \dots m$$

Take $$\sum_{i=1}^{m} \beta_i x_i = \sum_{i=1}^{m} \beta_i \sum_{j=1}^{n} \alpha_{ij} \alpha a_j = \sum_{j=1}^{n} \Big( \underbrace{\sum_{i=1}^{m} \alpha_{ij} \beta_j}_{*} \Big) a_j$$ → see

$$\sum_{i=1}^{m} \alpha_{ij} \beta_j = 0, \ \forall j = 1, 2, \dots\dots n$$

$$\sum_{i=1}^{m} \alpha_{ij} x_i = 0 \quad \text{as } n < m$$

(*) can be made 0 by non-trivial choice of $\beta_i$'s.

This implies $\{x_1, \dots x_m\}$ is lin d ⚡

**Defn**: A vector space is finite dimensional if it has a basis consist of finitely many vectors.

**Theorm**: In a finite dimensional vector space any basis has same number of elements.

**proof**: Let A, B be two basis of X with ~~#~~A, ~~#~~B cardinality from prev lemma

$\# A \leq \# B$ as B is spanning set
$\# B \leq \# A$ as A is spanning set.

$\Rightarrow \# A = \# B$
$\therefore$ cardinality of A = cardinality of B
dim X = cardinality
$= \# A$
↳ A is a basis

**Theorm**: (i) Any subspace of a finite dimensional vector space is finite dimensional

(ii) There exist a subspace Z of X s.t $Y \cap Z = \{\cdot\}$ and
$X = Y + Z = \{ y + z \mid y \in Y, z \in Z \}$
$\Rightarrow \dim(X) = \dim(Y) + \dim(Z)$

**proof** (i) If $\{y_1, y_2, \dots y_n\}$ lin ind in Y. Then by "the lemma" $\leq \dim X$

also by tutorial problem (2.4), $\dim Y \leq \dim X$ → see

(ii) let $\{\underbrace{y_1, y_2, \ldots y_m}, z_1, z_2, \ldots z_n\}$ be a basis of $X$.

vectors added to $Y$ to make a basis of $X$.

let $Z = \text{span}\{z_1, z_2, \ldots z_n\}$

$Y = \text{span}\{y_1, y_2, \ldots y_m\}$

$X = \text{span}\{y_1, y_2, \ldots y_m, z_1, z_2, \ldots z_n\}$

$Z \cap Y = \{\cdot\}$ as if $x \in Z \cap Y$

$\Rightarrow x \in Z$ and $x \in Y$

but since $\text{span}(Y) = \sum_{i=1}^{m} c_i y_i = x$

$\text{span}(Z) = \sum_{i=1}^{n} c_i' z_i = x$

so $x \in \text{span}(Y)$ and $x \in \text{span}(Z)$

$$\sum_{i=1}^{m} \alpha_i y_i - \sum_{j=1}^{n} \beta_j z_j = 0$$

as $\sum_{j=1}^{n} \beta_j z_j = \sum_{i=1}^{m} \alpha_i y_i$ $*$

$\Rightarrow \alpha_i = \beta_j = 0 \;\; \forall i, j$

$\therefore Z \cap Y = \{\cdot\}$

for any $\alpha \in X$, $\alpha = \text{span}\{y_1, y_2, \ldots y_m, z_1, z_2, \ldots z_n\}$

$= \sum c_i y_i + \sum c_j' z_j$

$\alpha = y' + z'$

$\Rightarrow X = Y + Z$

also $\dim X = \#\{y_1, y_2, \ldots y_m, z_1, z_2, \ldots z_n\}$

$\# X = m + n = \# Y + \# Z$

$\dim X = \dim Y + \dim Z$

If $X_1, X_2, \ldots X_n$ are subspaces of $X$ then

$X_1 + X_2 + \cdots X_n = \{x_1 + x_2 + \cdots x_n \mid x_i \in X_i,\ i = 1, 2, \ldots n\}$

if pairwise disjoint:

$\dim(X_1 + X_2 + \cdots X_n) = \sum_{i=1}^{n} \dim(X_i)$

<u>Note</u>: $Y$ – subspace of $X$

<u>defn</u>: $x_1 \equiv x_2 \pmod{Y} \Leftrightarrow x_1 - x_2 \in Y$

Sre: Show that congruence modulo $Y$ defines an equivalance relation in $X$. → do

Claim: $[x] = x + Y = \{x + y \mid y \in Y\}$

equivalance class

all elements equivalent to $x$

as for $x' \in [x]$

$x' = x + y$ for some $y \in Y$

$x' - x = y \in Y$

$\therefore x' \equiv x \pmod{Y}$

also if $z \equiv x \pmod{Y}$

$\Rightarrow z - x \in Y$

$\Rightarrow z - x = y$, for some $y \in Y$

$\Rightarrow z = x + y$

$\Rightarrow z \in [x]$

Define: (i) $(x_1 + Y) + (x_2 + Y) = (x_1 + x_2) + Y$

(ii) $\alpha(x + Y) = \alpha x + Y$

$X/Y = \{x + Y \mid x \in X\}$ becomes a vector space

Theorm: let $Y$ be a subspace of finite dimentional space $X$.

then $\dim(X/Y) = \dim(X) - \dim(Y)$

Proof: let $\{y_1, y_2, \ldots y_m\}$ be basis of $Y$.

and let

$\{x_1, x_2 \ldots x_n, y_1, y_2, \ldots y_m\}$ be basis of $X$.

now $\{x_i + Y \mid i = 1, 2, \ldots n\}$ forms a basis for $X/Y$ then we are done

① spans $X/Y$

② lin ind

lin ind: if we take linear combination then

$$\sum_{i=1}^{n} \alpha_i (x_i + Y) = 0 + Y$$

$$\Rightarrow \sum_{i=1}^{n} (\alpha_i x_i + Y) = 0 + Y$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i x_i + Y = 0 + Y$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i x_i - 0 \in Y$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i x_i \in Y$$

then $\sum_{i=1}^{n} \alpha_i x_i = \sum_{i=1}^{m} \beta_i y_i = y$

if non-zero then they don't form basis of $X$

$\Rightarrow y = 0$

$\Rightarrow \sum_{i=1}^{n} x_i \alpha_i = 0 \Rightarrow \alpha_i = 0, \forall i \in 1, 2, \ldots n$ (subset of lin ind set)

$\therefore$ lin ind

spans $X/Y$ : $\{x_i + Y \mid i = 1, 2, \cdots n\}$

for any $x \in X$,

$$x = \sum_{i=1}^{n} \alpha_i x_i + \sum_{j=1}^{n} \beta_j y_j$$

$$x + Y = \sum_{i=1}^{n} \alpha_i x_i + Y$$

$$x + Y = \underbrace{\sum \alpha_i (x_i + Y)}$$

any $x+Y$ can be written as lin comb of $x_i + Y$

$\therefore$ spans $X/Y$

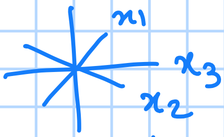
$\therefore \dim(X/Y) = \dim X - \dim Y$

$19^{th}$ Aug : **conventions** : $X$ - vector space

$X_1, X_2, \ldots X_n \subseteq X$ — subspaces of $X$

$$\dim(X_1 \oplus X_2 \oplus \cdots) = \sum_{i=1}^{n} \dim(X_i)$$

if every vector in $x \in X$ can be written uniquely as $x = x_1 + x_2 + \cdots x_n$, $x_i \in X_i$




counter example of why not correct (original) as $\sum \dim(x_i) = 3$ but $\dim X = 2$

**morphisms** :

**Defn**: (linear map) let $X$ and $Y$ be vector spaces over $F$. A map $T: X \to Y$ is a linear map / linear transformation / linear operator if

(i) $T(x_1 + x_2) = T(x_1) + T(x_2)$, $\forall\, x_1, x_2 \in X$

(ii) $T(\alpha x) = \alpha T(x)$, $\alpha \in F$, $x \in X$.

Note : If $Y = F$, then $T$ is said to be a linear functional.

$$T(\underbrace{\alpha_1 x_1 + \alpha_2 x_2}_{\text{and so on}} + \cdots \alpha_n x_n) = \underbrace{\alpha_1 T(x_1) + \alpha_2 T(x_2) + \cdots}_{\text{to get the this}} \alpha_n T(x_n)$$

and so on . . . to get the this.

**Examples** : (i) $X = C([0,1]) = \{ f: [0,1] \to \mathbb{R}, f \text{ is cont} \}$

Fix $\lambda \in [0,1]$ $\quad T: C([0,1]) \to \mathbb{R}$

$T(f) = f(t)$, $\forall\, f \in C([0,1])$

$\rightsquigarrow f, g \in C([0,1])$

$$T(f+g) = (f+g)(t) = f(t) + g(t) = T(f) + T(g)$$

for $\alpha \in \mathbb{R}$ and $f \in C([0,1])$, $T(\alpha f) = (\alpha f)(t) = \alpha f(t) = \alpha T(f)$

(ii) let $X$ = set of polynomials of degree $\leq n$

$$\frac{d}{dx} : X \to X$$

$$f \to \frac{df}{dx}$$

**Null space / kernal** : $T: X \to Y$

$\ker(T) = N_T = \{ x \in X \mid T(x) = 0 \}$ = Null space

ker(T) is a subspace :

for $x, y \in \ker(T)$

$$T(\lambda x + y) = T(\lambda x) + T(y) = 0$$

$\therefore\ \alpha x + y \in N_T$

$\therefore$ ker(T) is a subspace of $X$.

**Range of T** : $\mathrm{Ran}\,T = R_T = \{T(x) \mid \forall x \in X\}$

**RanT is a subspace** : $\alpha \in F$, $T(x_1) \in R_T$, $T(x_2) \in R_T$

now $\alpha T(x_1) + T(x_2)$
$= T(\alpha x_1) + T(x_2)$
$= T(\alpha x_1 + x_2)$

as $T(\alpha x_1 + x_2) \in R_T$

$\mathrm{Ran}\,T = R_T$ is a subspace of $Y$.

**Isomorphism**: let $X$ and $Y$ be vector spaces over $F$.
A linear map $T: X \to Y$ is an isomorphism if $T$ is a bijective map.

**Lemma**: A linear map $T: X \to Y$ is injective $\Leftrightarrow \ker T = N_T = \{0\}$

**proof**: ($\Rightarrow$) Trivial

($\Leftarrow$) for $x_1 \neq x_2 \in X$ suppose
$T(x_1) - T(x_2) = T(x_1 - x_2) = 0$
$\Rightarrow x_1 - x_2 \in N_T = \{0\}$
$\Rightarrow x_1 = x_2$ $\ast$

**Defn**: two vectorspaces are isomorphic, $\exists$ linear map that is isomorphism.

**Theorem**: let $X$ and $Y$ be finite dimensional vector spaces over $F$.
Then $X$ is isomorphic to $Y \Leftrightarrow \dim X = \dim Y$.

**Note**: $F^m$ is $m^{th}$ degree subspace of $F$ (as $\dim F^m = m$)

**proof**: ($\Leftarrow$) let $\{x_1, x_2, \ldots x_m\}$ be basis for $X$ and $\{y_1, \ldots y_m\}$ be a basis for $Y$.

Define $T: X \to Y$

$$T\left(\sum_{i=1}^{m} \alpha_i x_i\right) = \sum_{i=1}^{m} \alpha_i y_i, \quad \forall \alpha_i \in F, \; x_i \in X.$$

① **well defined**:

$\sum_{i=1}^{m} a_i x_i$ is unique then

$\sum \alpha_i y_i$ is also unique.

② **linear map**: Trivial (as $T(\sum \alpha_i x_i + \sum \beta_i x_i)$, $T(\alpha \sum \alpha_i x_i)$)

③ **one-one**: now $N_T = \{x \in X \mid T(x) = 0\}$

if $T\left(\sum_{i=1}^{m} \alpha_i x_i\right) = 0$, then

$\sum \alpha_i y_i = 0$
$\Rightarrow \alpha_i = 0 \quad \forall i \in \{1, 2, \ldots n\}$

$\therefore T(0) = 0$ only $T(0) = 0$

$\therefore \{0\} = N_T$

as $N_T = \{0\} \Rightarrow T$ is one-one.

④ onto: $R_T = Y$ (follows from fact, $y_i \in R_T$)

$\therefore$ $X$ is isomorphic to $Y$.

($\Rightarrow$) let $T: X \to Y$ be an isomorphism

Suppose $\{x_1, x_2, \dots, x_n\}$ is a basis for $X$.

<u>claim</u>: $T(x_1), \dots, T(x_n)$ is a basis for $Y$.

suppose $\sum_{i=1}^{n} \alpha_i T(x_i) = 0$

$\Rightarrow T(\sum_{i=1}^{n} \alpha_i x_i) = 0$

as $N_T = \{0\}$

$\Rightarrow \sum_{i=1}^{n} \alpha_i x_i = 0$

$\Rightarrow \alpha_i = 0, \forall i = 1, 2, \dots n$

$\therefore$ $\{T(x_1), \dots T(x_n)\}$ is <u>lin ind</u>.

as Ran $T = Y$

Take $y \in Y$. Then $\exists x \in X$ s.t $T(x) = y$. since $\{x_1, \dots x_n\}$ is a basis for $X$,

$x = \sum_{i=1}^{n} \beta_i x_i$

$T(\sum_{i=1}^{n} \beta_i x_i) = y$

$\Rightarrow \sum_{i=1}^{n} \beta_i T(x_i) = y$

$\forall y \in Y$

$\therefore$ span $\{T(x_1), T(x_2) \dots T(x_n)\} = Y$

$\therefore$ $\{T(x_1), T(x_2) \dots T(x_n)\}$ is a basis of $Y$.

$\therefore$ $\dim X = \dim Y$

<u>Exercise</u>: let $X$ be an $n$-dimensional vector space over $F$. Find an isomorphism, between $X^m$ and $F^m$! $\to$ done (see down)

<u>Example</u>: (iii) $\begin{bmatrix} c_1 c_2 \dots c_n \end{bmatrix}_{m \times n}$ $c_i \in \mathbb{R}^m$ this is a linear map from

$\begin{bmatrix} c_1 c_2 \dots c_n \end{bmatrix} : \mathbb{R}^n \to \mathbb{R}^m$

as $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ is $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$

$$\begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}_{m \times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} = x_1 c_1 + x_2 c_2 + \dots x_n c_n \in \mathbb{R}^m$$

<u>Exercise</u>: matrix multiplication defines a linear map $\to$ done (see down)

**Theorem:** let $X, Y$ be finite dimensional vector spaces over $F$. If $T: X \to Y$ is a linear map, then

$$\dim X = \underbrace{\dim(N_T)}_{\text{nullity}} + \underbrace{\dim(R_T)}_{\text{rank}}$$

**proof:** $\tilde{T} : X/N_T \to Y$

$\tilde{T}(x + N_T) = T(x)$

$x_1 + N_1 = x_2 + N_2 \Leftrightarrow x_1 - x_2 \in N_T$

this means map is well defined.

$T(x_1 - x_2) = 0$

$\Rightarrow T(x_1) = T(x_2)$

$\tilde{T}(x_1 + N_T) = \tilde{T}(x_2 + N_T)$

$\therefore \tilde{T}$ is well defined.

$$\begin{aligned} \tilde{T}((x_1 + N_T) + (x_2 + N_T)) &= \tilde{T}(x_1 + x_2 + N_T) \\ &= T(x_1 + x_2) \\ &= T(x_1) + T(x_2) \\ &= \tilde{T}(x_1 + N_T) + \tilde{T}(x_2 + N_T) \end{aligned}$$

and $\lambda \tilde{T}(x_1 + N_T) = \lambda T(x_1) = T(\lambda x_1) = \tilde{T}(\lambda x_1 + N_T)$

$\therefore \tilde{T}$ is a linear map.

$\tilde{T}(x + N_T) = 0$

$\Rightarrow T(x) = 0$

$\Rightarrow x \in N_T$

$\Rightarrow x + N_T = N_T$

$\therefore \ker(\tilde{T}) = \{0\}$ $\tilde{T}$ is one-one

$$\tilde{T} : X/N_T \to R_{\tilde{T}} \subseteq Y, \quad R_{\tilde{T}} = R_T$$

$\tilde{T} : X/N_T \to R_T$ is also one-one and surjective to $R_T$

$$\dim(X/N_T) = \dim(R_T)$$

$$\Rightarrow \dim(X) - \dim(N_T) = \dim(R_T)$$

$$\Rightarrow \dim(X) = \dim(N_T) + \dim(R_T)$$

Exercise : let $X$ be an $n$-dimensional vector space over $F$. Find an isomorphism, between $X^m$ and $F^m$.

let Basis of $X$ be $\{x_1, x_2, \ldots x_n\}$ as it is $n$-dimentional
now,

$X^m$ is $m^{th}$ degree subspace of $X$ and
$F^m$ is $m^{th}$ degree subspace of $F$.

let Basis of $X^m = \{x_1, x_2, \ldots x_m\}$

let $x_m \in X^m = \sum_{i=1}^{m} \alpha_i x_i$

and let $T: X^m \to F^m$

$$x_m \to \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} \in F^m$$

now, $$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{m\times 1} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{m\times 1} + \cdots \alpha_m \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}_{m\times 1}$$

$$\text{Basis of } F^m = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

as it ① span $F^m$
② lin ind

$\therefore$ as $\deg F^m = \deg X^m$
$\Rightarrow$ $F^m$ is isomorphic to $X^m$.

$T: X^m \to F^m$

$$x_m = \sum_{i=1}^{m} \alpha_i x_i \qquad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}$$

$T$ is a linear map
as

① $T(x_{m_1} + x_{m_2})$
$= \begin{pmatrix} \alpha_1 + \beta_1 \\ \vdots \\ \alpha_m + \beta_m \end{pmatrix}$
$= T(x_{m_1}) + T(x_{m_2})$

② $T(\alpha x_m) = \alpha T(x_m)$

① well defined :
as for $\sum \alpha_i x_i = \sum \beta_i x_i$
$\Rightarrow \alpha_i = \beta_i$
$\therefore T(\sum \alpha_i x_i)$
$= T(\sum \beta_i x_i)$

② one-one :
$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$
$\Rightarrow \alpha_i = \beta_i$
$\Rightarrow \sum \alpha_i x_i = \sum \beta_i x_i$

③ onto :
for any set $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} \in F$, $\exists \sum \alpha_i x_i = x_m$
$\therefore$ onto

$\therefore$ $T: X^m \to F^m$ is an isomorphism

Exercise : matrix multiplication defines a linear map

$$\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}_{m\times n} : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{defined as:}$$

$$\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

$$= x_1 c_1 + x_2 c_2 + \cdots x_n c_n$$

linear map :

① $$\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} = (x_1+y_1)c_1 + \cdots (x_n+y_n)c_n$$

$$= \begin{bmatrix} c \end{bmatrix} x + \begin{bmatrix} c \end{bmatrix} y$$

② $$\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix} = \alpha x_1 c_1 + \cdots \alpha x_n c_n$$

$$= \alpha (x_1 c_1 + \cdots x_n c_n)$$

$$= \alpha \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

22nd Aug :

**Theorem** : (Rank nullity theorem) let $X$ be a finite dimensional vector space, and $V$ be a vector space over $F$. If $T: X \to V$ is a linear map then $\dim X = \dim(N_T) + \dim(R_T)$

group theory eq:

$$T: X \to V$$
$$\tilde{T}: X/N_T \to V$$
$$X/N_T \cong R_{\tilde{T}} = R_T$$
$$\therefore \dim(X) - \dim(N_T) = \dim(R_T)$$

**cor**: let $T: X \to V$ be a linear map (I) If $\dim(V) < \dim(X)$ then $\exists$ a non-zero vector $x$ s.t $T(x) = 0$.

finite vector space

as $T(0) = 0$
$T(\lambda x) = \lambda T(x), \forall \lambda \in F$
and if $\lambda = 0$
$T(0) = 0$ (always)

$m < n$ $\sum_{j=1}^{n} a_j x_j = 0$ $x_j \in \mathbb{R}^m$
$A = \left( \quad \right)_{m \times n} : \mathbb{C}^n \to \mathbb{C}^m$

(II) If $\dim(V) = \dim(X)$ and if $N_T = \{0\}$, then $R_T = V$
injective $\Rightarrow$ surjective $\Rightarrow$ bijective

(III) If $\dim(V) = \dim(X)$ and $R_T = V$, then $N_T = \{0\}$
surjective $\Rightarrow$ injective $\Rightarrow$ bijective

**proof** : (I) $\dim(N_T) = \dim(X) - \dim(R_T)$
$\geq \dim(X) - \dim(V) > 0$
$\Rightarrow \dim(N_T) > 0$

(II) $\dim(N_T) = 0$
$\dim(X) = \dim(R_T) = \dim(V)$
$\Rightarrow R_T = V$
as $R_T \subseteq V$

(III) $\dim(V) = \dim(X)$
and $R_T = V \Rightarrow \dim(R_T) = \dim(X)$
$\Rightarrow \dim(N_T) = 0$
$\Rightarrow N_T = \{0\}$

**Note** : now, $Ax = b, A \in M_{n \times n}$
if $Ax = 0$ has only the trival soln then $N_A = \{0\}$ then it means $\dim(N_A) = 0$ then $Ax = b$ will have a unique solution.

as $Ax = b$
↳ linear map
$A: \mathbb{R}^n \to \mathbb{R}^n$
and as $b \in \mathbb{R}^n$ is in $R_A$
$R_A = \mathbb{R}^n$
$\therefore Ax = b$ will have a unique sol (one-one, onto)

**Note** : $\mathcal{L}(X,V) = \{T: X \to V \mid T \text{ is linear}\}$
$\hookrightarrow$ vector space

* $\mathcal{L}(X,V)$ is a vector space

**proof** : $T_1 + T_2 : X \to V$
$(T_1+T_2)(x) = T_1(x) + T_2(x)$

$\lambda T : X \to V$
$(\lambda T)(x) = \lambda T(x), \forall x \in X$

**Theorm** : let $\{x_1, x_2, \dots x_n\}$ be a basis for $X$. There is an isomorphism

$$\pi : \mathcal{L}(X,V) \to V \times V \times V \dots V$$
$$T \longmapsto (T(x_1), T(x_2) \dots T(x_n))$$

**proof** : ① well-defined
② linear
③ injective
④ surjective

} Needed for isomorphism

**well-defined** : $T_1 = T_2$
$T_1(x_1) = T_2(x_1)$
$T_1(x_i) = T_2(x_i)$
$\therefore$ $\pi$ is a well-defined map.

**linear** : $\pi(\lambda T_1 + T_2) = ((\lambda T_1 + T_2)(x_1), \dots (\lambda T_1 + T_2)(x_n))$
$= (\lambda T_1(x_1) + T_2(x_1), \dots \lambda T_1(x_n) + T_2(x_n))$
$= \lambda (T_1(x_1), T_1(x_2), \dots T_1(x_n))$
$+ (T_2(x_1), \dots T_2(x_n))$

$= \lambda \pi(T_1) + \pi(T_2)$

**injective** : If $T \in \mathcal{L}(X,V)$, such that $T(x_i) = 0, \forall i = 1, 2, \dots n$

then $T(x) = T(\sum_{i=1}^{n} \lambda_i x_i)$
$= \sum_{i=1}^{n} \lambda_i T(x_i)$
$= 0 \quad , \forall x \in X$
$\therefore T \equiv 0$

so if $T(x_i) = 0 \ \forall i$
then $T \equiv 0$ (it is a zero map)

$\therefore N_\pi = \{0\}$
$\hookrightarrow$ all linear zero maps.

**surjective** : let $(v_1, \dots v_n) \in V \times V \times \dots V$
define $T_v : X \to V$ by

$T_v(\sum_{i=1}^{n} \alpha_i x_i) = \sum_{i=1} \alpha_i v_i$ where $v_i = T(x_i)$

① $T_v$ is well defined
② $T_v$ is linear map
as $T_v(\alpha x + y) = \alpha T_v(x) + T_v(y)$

now, $(T_V(x_1), T_V(x_2), \ldots \ldots T_V(x_n)) = (V_1, V_2, \ldots V_n)$

$\alpha_1 = 1$, $\alpha_i = 0$ for $i \neq 1$; $\alpha_2 = 1$, $\alpha_i = 0$ for $i \neq 2$

$\therefore$ Isomorphism

<u>cor</u>: let $T \in \alpha(X,V)$ and $\{x_1, x_2, \ldots x_n\}$ is basis then (I) $T$ is injective $\Leftrightarrow \{T(x_1), \ldots T(x_n)\}$ is lin ind set of vetor in $V$.

(II) $T$ is surjective $\Leftrightarrow$ span $\{T(x_1), T(x_2), \ldots T(x_n)\} = V$

So here see $T$ is bijective $\Leftrightarrow$ Basis $V = \{T(x_1), \ldots T(x_n)\}$

<u>sketch of proof</u>: (I) $T(\sum \alpha_i x_i) = 0 \Leftrightarrow \sum \alpha_i T(x_i) = 0$

as $N_T = \{0\}$

only if

$\sum \alpha_i x_i = 0$

$\Rightarrow \alpha_i = 0$

(II) $T$ is surjective,

$R_T = V$ any $v \in V$

$v = T(x)$

$\Rightarrow v = T(\sum \beta_i x_i)$

$\Rightarrow v = \beta_i T(\sum x_i)$

or span has to be whole space.

<u>proof</u>: ($\Rightarrow$) $T$ is one-one

if $T(\sum \alpha_i x_i) = 0$ $\alpha_i = 0$

$\Rightarrow \sum \alpha_i T(x_i) = 0$

$\Rightarrow \alpha_i = 0 \therefore$ lin ind

($\Leftarrow$) if $\sum \beta_i T(x_i)$ is lin ind

$\sum \beta_i T(x_i) = 0$

$\Rightarrow \beta_i = 0$

also $\sum \beta_i T(x_i)$

$= T(\sum \beta_i x_i)$

$0 = T(0)$

<u>Proof</u>:

($\Rightarrow$) $T$ is surjective

then any $v \in V$

$v = T(x)$

$v = T(\sum \alpha_i x_i)$

$\Rightarrow v = \sum \alpha_i T(x_i)$

$\therefore$ span.

($\Leftarrow$) if $T(x_i)$ span

then $\sum \beta_i T(x_i) = T(\sum \beta_i x_i)$, $\forall x \in X$

$= T(x)$ $\therefore V = R_T$

<u>defn</u>: $X$ - n dimentional vector space

$X' = \alpha(X, f)$ - dual space

<u>Note</u>: $X' \cong F^n$, $\dim(X') = \dim(F^n) = n$

basis of $F^n$:

$e_i = (0, \ldots 0, 1, 0, \ldots 0)$ basis for $F^n$.

$i^{th}$ place

<u>defn</u>: let $Y \subseteq X$ be a subspace of $X$

$Y^\perp = \{\gamma \in \alpha(X, f) \mid \gamma(y) = 0, \forall y \in Y\}$ → <u>Annihilator of $Y$</u>

$Y^\perp \subseteq X^\perp$

$$(\alpha\gamma_1 + \gamma_2)(y) = \alpha\gamma_1(y) + \gamma_2(y)$$
$$= \alpha \cdot 0 + 0$$
$$= 0 \quad , \forall y \in Y$$

$\therefore Y^\perp \subseteq X^\perp$

Theorm: let $Y$ be a subspace of a finite dimensional vector space $X$. Then $Y^{\perp} \cong (X/Y)' \rightarrow$ dual space

and $\dim(X) = \dim(Y) + \dim(Y^{\perp})$

$$(X/Y)' = \alpha(X/Y, F)$$

$$(X/Y)' \cong F^{\dim(X)-\dim(Y)}$$

$Y^{\perp} \rightarrow (0,0,0,\ldots * * \ldots)\ F^{n}$

for $\alpha(X,V) \rightarrow (\underbrace{T(x_1), T(x_2) \ldots}_{\text{Basis of } Y^{\perp}} T(x_n))$

then $(\underbrace{0,0,\ldots}_{m} T(x_1), T(x_2) \ldots T(x_{n-m}))\ F^{m}$

$\dim(X) = \dim(Y^{\perp}) + \dim(Y)$

**Proof**: Define a map

$$\Gamma : Y^{\perp} \longrightarrow (X/Y)'$$

$$\tau \longmapsto \sigma$$

s.t $\sigma(x+Y) = \tau(x)$

$$x_1 + Y = x_2 + Y$$
$$\Leftrightarrow x_1 - x_2 \in Y$$
$$\Leftrightarrow \tau(x_1 - x_2) = 0$$
$$\Leftrightarrow \tau(x_1) = \tau(x_2)$$

well defined and one-one

26th Aug : Quiz – wednesday – till last class (not this)

**Theorem**: Let $X$ be a finite dimensional vector space and $Y \subseteq X$ be a subspace. Then

$$\dim(Y^\perp) = \dim((X/Y)')$$

proof: claim: there is an isomorphism b/w $Y^\perp$ and $(X/Y)'$.

$\Gamma : Y^\perp \to (X/Y)'$

$\gamma \to \Gamma(\gamma)$ where

$\Gamma(\gamma) : X/Y \to F$ (F is a field, wlog)

linear map

defined by

$\Gamma(\gamma)(x+Y) = \gamma(x) \quad \forall x \in X$

① well-defined —— trivial

② linear: $\Gamma(\gamma_1+\gamma_2)(x+Y) = (\gamma_1+\gamma_2)(x) = \gamma_1(x) + \gamma_2(x)$
$= \Gamma(\gamma_1)(x+Y) + \Gamma(\gamma_2)(x+Y)$

$\Rightarrow \Gamma(\gamma_1+\gamma_2) = \Gamma(\gamma_1) + \Gamma(\gamma_2)$

$\Gamma(\lambda\gamma)(x+Y) = (\lambda\gamma)(x+y) = \lambda\gamma(x+y)$
$= \lambda\Gamma(\gamma)(x+Y)$

③ one-one: if $\Gamma(\gamma) = 0$, then $\Gamma(\gamma)(x+Y) = 0$, $\forall x \in X$

$\Rightarrow \gamma(x) = 0, \forall x \in X$

$\Rightarrow \gamma = 0$

$\therefore N_\Gamma = \{0\}$

$\Rightarrow \Gamma$ is one-one

④ onto: $X \xrightarrow{\pi} X/Y \xrightarrow{\theta} F$

$x \longmapsto x+Y$

vanishes on $Y$ — vanishes on $Y$

$\pi$: linear surjection → trivial $(\pi(x) = x+Y)$

$\theta \circ \pi : X \to F$ → composition of linear maps is linear

$\theta \circ \pi(x_1+x_2) = \theta(\pi(x_1+x_2))$

$= \theta(\pi(x_1) + \pi(x_2))$

$= \theta\circ\pi(x_1) + \theta\circ\pi(x_2)$

also $\theta\circ\pi(\lambda x) = \theta\circ(\lambda\pi(x)) = \lambda\,\theta\circ\pi(x)$

Also vanishes at $Y$, as $\pi$ kills $y \in Y$.

$\Rightarrow \theta\circ\pi \in Y^\perp$

need to show $\Gamma(\theta\circ\pi) = \theta$ then done as $\theta \in (X/Y)'$

$\Gamma(\theta\circ\pi)(x+Y) = \theta\circ\pi(x)$
$= \theta(\pi(x))$
$= \theta(x+Y), \forall x \in X$

$\therefore$ onto

**Transpose :** $T: X \longrightarrow Y$

$\rightsquigarrow T': Y' \longrightarrow X'$ (Transpose of $T$)

$$T'(Q)(x) = Q(T(x)), \quad Q \in Y', \; x \in X$$

$$X \xrightarrow{T} Y \xrightarrow{Q} F \qquad = Q \circ T$$

$T: X \to Y$

$Q: Y \to F$

then $Q \circ T : X \longrightarrow F$

$\tau \rightarrow$ cont map

$[a, b] \xrightarrow{\tau} [c, d]$

$\rightsquigarrow T: C([c,d]) \longrightarrow C([a,d])$

$f \longmapsto f \circ \tau$

**Exercise :** show that if $\tau$ is surjective then $T$ is injective. Here $T: f \to f \circ \tau$

$\tau$ is sujetive $R_T' = X'$

but $\dim(X') = \dim(R_T') + \dim(N_T')$

$\Rightarrow N_T' = \{0\}$

this theorm can be used $\Rightarrow$ one-one

**Note :** $T'(Q)(x) = Q(T(x))$, $Q \in Y'$, $x \in X$

$\underbrace{\phantom{Q(T(x))}}$ composing linear maps

(see prev auto proof)

**Theorm :** let $T: X \longrightarrow Y$ be a linear map, show that $X$ and $Y$ are finite dimensional vector spaces, then $R_T^{\perp} = N_{T'}$

**proof :** For $Q \in R_T^{\perp}$

$\Leftrightarrow Q(y) = 0, \; \forall y \in R_T$

$\Leftrightarrow Q(T(x)) = 0, \; \forall x \in X$

$\Leftrightarrow (T'(Q))(x) = 0, \; \forall x \in X$

$\Leftrightarrow T'(Q) \equiv 0$

$\Leftrightarrow Q \in N_{T'}$

for $T'(Q)(x)$

$= \underline{Q \circ T} = 0$

$Q$ has to vanish on range of $T$

$T: X \to Y$

So $Q \in R_T^{\perp}$

**Theorm :** let $X$ and $Y$ be f.d vector spaces and $T: X \longrightarrow Y$ is a linear map then $\dim(R_T) = \dim(R_{T'})$

**proof :**

$\dim(Y') = \dim(R_{T'}) + \dim(N_{T'})$ — By Rank-nullity theorm

$\dim(Y) = \dim(R_T) + \dim(R_T^{\perp})$ — By theorm we proved today

as

$\dim(Y) = \dim(Y')$ $\rightarrow$ see

$\dim(R_T^{\perp}) = \dim(N_{T'})$

then

$\dim(R_T) = \dim(R_{T'})$

**Example :**

$X \longrightarrow X' \longrightarrow X''$ (dual space, dual space)

$\dim(X) = \dim(X') = \dim(X'')$

$X'' \cong X$ (as $X \cong X' \cong X''$)

**Dual space of $\mathbb{R}^n$ :**

$\mathbb{R}^n$ and $(\mathbb{R}^n)'$

$$(\mathbb{R}^n)' = \{ T: \mathbb{R}^n \to \mathbb{R} \mid T \text{ is linear} \}$$

$$e_i = (\underbrace{0, 0, \ldots 1}_{i\text{th}}, \underbrace{0, 0 \ldots 0}_{n-i}) \quad \text{bases of } \mathbb{R}^n$$

$Q \in (\mathbb{R}^n)'$

$Q \longleftrightarrow (Q(e_1), Q(e_2), \ldots Q(e_n))$

$Q(\sum \alpha_i e_i) = \sum \alpha_i Q(e_i)$

$(\alpha_1, \alpha_2, \ldots \alpha_n) \longmapsto \sum \alpha_i Q(e_i)$

$(Q(e_1), \ldots Q(e_n)) \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \sum_{i=1}^{n} \alpha_i Q(e_i)$

row vector

uniquely determines

$T = \left[ T_{ij} \right]_{n \times m} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$T' : (\mathbb{R}^n)' \longrightarrow (\mathbb{R}^m)'$

$T'(Q)(x) = Q(T(x)), \forall Q \in Y', x \in X$

here $T'(Q)(x) = Q(T(x)), \forall Q \in (\mathbb{R}^n)', x \in \mathbb{R}^m$

For $Q = (Q_1, \ldots Q_n) \in (\mathbb{R}^n)'$ and

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$$

$$Q\left(T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}\right) = (Q_1 \; Q_2 \ldots Q_n) \begin{pmatrix} \sum_{j=1}^{m} T_{1j} x_j \\ \vdots \\ \sum_{j=1}^{m} T_{nj} x_j \end{pmatrix}$$

$$= \sum_{l=1}^{n} Q_l \sum_{j=1}^{m} T_{lj} x_j = \sum_{j=1}^{m} \left( \sum_{l=1}^{n} T_{lj} Q_l \right) x_j$$

Note: $(T^t)_{ij} = T_{ji}$

$$= \sum_{j=1}^{m} \left( \sum_{l=1}^{n} T^t_{jl} Q_l \right) x_j$$

$$= \left( \sum_{1}^{n} T^t_{1l} Q_l, \sum_{1}^{n} T^t_{2l} Q_l, \ldots \sum_{1}^{n} T^t_{ml} Q_l \right) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

see $\rightarrow$ $= T^t(Q) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$

see $\rightarrow$ $\Rightarrow T' : (\mathbb{R}^n)' \longrightarrow (\mathbb{R}^m)'$

$\underline{T' = T^t}$

29th Aug : Recap:

Proved : $T: \mathbb{R}^n \to \mathbb{R}^m$, $T': \mathbb{R}^m \to \mathbb{R}^n$ if $T'$ is a transpose map then $T' = T^t$ matric transpose

**Defn**: The column rank of a matrix $T \in M_{n\times n}(F)$ is the dimension of the span $\{C_1, \dots, C_n\}$ then $T = [C_1, C_2, \dots, C_n]$.

The row rank of T is dimension of the span $\{R_1, R_2, \dots, R_m\}$, then
$$T = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}$$

- column rank of $T: \mathbb{R}^n \to \mathbb{R}^m$ $T = \dim(R_T)$

$$[c_1, c_2, \dots c_n]_{m\times n} \qquad R_T = \left\{ [c_1, c_2, \dots c_n] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| x_i \in F \right\}$$
$$= \left\{ \sum_{i=1}^{n} x_i c_i \middle| x_i \in F \right\} = \text{span}\{c_1, c_2, \dots c_n\}$$

Row rank of $T$ = column rank of $T^t = \dim(R_{T^t})$

**Theorm**: For $T \in M_{m\times n}(\mathbb{F})$, column rank of T = row rank of T.

$T \in M_{m\times n}(\mathbb{F})$ $\quad T: \mathbb{R}^n \to \mathbb{R}^m$

$\mathbb{R}^n$: Row space of T and Null space of T — subspace of $\mathbb{R}^n$ $\Downarrow$ $\boxed{n-r}$

$\mathbb{R}^m$: column space of T $= \boxed{r}$, Null space of $T^t$ — subspace of $\mathbb{R}^m$ $\Downarrow$ $\boxed{m-r}$

$$\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{T^t} \begin{bmatrix} 1 & 0 & 0 \\ * & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\Updownarrow$
$T$ $\quad$ why is rank of T and $T^t$ same?
$ET = R$

$T \leftrightarrow [\quad]$
$Tx = E^{-1}Rx = 0$

$\text{Span}\{R_1 T, \dots R_m^T\}$
$\text{Span}\{\lambda R_1 T, \dots R_m T\}$ same span T and T'

$N(T^t) = \{ y \in \mathbb{R}^m , T^t y = 0 \}$
$= \{ y \in \mathbb{R}^m \mid y^t T = 0 \}$

$$(y_1, \dots) [\quad] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$N(T) = \{ x \in \mathbb{R}^n \mid Tx = 0 \}$

$V \cong \mathbb{F}^n$

Let $\{v_1, \ldots, v_n\}$ be ordered basis

then $T : V \longrightarrow \mathbb{F}^n$

by $T(v_i) = e_i, \ \forall i = 1, 2, \ldots n$

$e_i = (0, \ldots, 1, \ldots)$

$$T\left(\sum_{i=1}^{n} \alpha_i v_i\right) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$\mathbb{R}^n \longleftarrow V \longrightarrow W \longrightarrow \mathbb{R}^m$$

Define $\pi : \mathcal{L}(V, W) \longrightarrow M_{m \times n}(\mathbb{F})$

$T \longmapsto B_W \circ T \circ B_V^{-1}$

then $\pi$ is an isomorphism

$\alpha S + T \longrightarrow B_W \circ (\alpha S + T) \circ B_V^{-1}$

$B_W \circ (\alpha S) \circ B_V^{-1} +$

$B_W \circ T \circ B_V^{-1}$

$\sim \pi$ is linear

Doubt (ask prof)

Injectivity: If $B_W \circ T \circ B_V^{-1} = 0$

then for any $x \in V$, $B_W T \circ B_V^{-1}(B_V x) = 0$

$\Rightarrow B_W(T(x)) = 0$

$\Rightarrow T(x) = 0$

Also for this theorm see above class proof for $\dim(R_T) = \dim(R_T')$ as $R_T' = R_{T^t}$

column rank = row rank

Surjectivity: For any $M \in M_{m \times n}(\mathbb{F})$

define $T : B_W^{-1} \circ M \circ B_V : V \longrightarrow W$

and $\pi(T) = B_W \circ (B_W^{-1} \circ M \circ B_V) \circ B_V^{-1} = M$

$$\begin{array}{c} c_1 \ c_2 \ldots c_n \\ \begin{bmatrix} | & | & & | \\ \end{bmatrix}_{m \times n} \end{array} \qquad T(e_1)$$

$$B_W \circ T \circ B_V^{-1}(e_i) = B_W \circ T(v_i) = B_W\left(\sum_{m} A_{ji} w_j\right) = (A_{1i}, A_{2i}, \ldots, A_{mi})$$

2nd sept :

$T : X \to X$

The matric rep of $T$ w.r.t a basis $x = \{x_1, \dots x_n\}$ is $A \in M_{m\times n}(\mathbb{C})$

$$T(x_\ell) = \sum_{i=1}^{n} A_{i\ell}\, x_i$$

as linear combinations of $x_i$'s

matric rep of $T$

$$A = \begin{array}{c} \begin{bmatrix} & A_{1\ell} & \\ & A_{2\ell} & \\ & \vdots & \\ & A_{n\ell} & \end{bmatrix} \\ \ell^{th} \text{ column} \end{array}$$

Trick: 
$$\begin{array}{c|c} & T(v_1)\; T(v_2) \dots T(v_n) \\ \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{matrix} & \begin{bmatrix} \quad & \quad \\ & \\ & \end{bmatrix} \end{array}$$

$T : V \to V$ Basis $= \{v_1, \dots v_m\}$; Basis $= \{v_1, \dots v_n\}$

If $B$ is a matric represtation of $T$ w.r.t another matric then

$y = \{y_1, y_2, \dots y_n\}$

$$\mathbb{F}^n \xleftarrow{B_y} X \xrightarrow{T} X \xrightarrow{B_y} \mathbb{F}^n$$
$B_y^{-1}$

$B_y(y_i) = e_i$ ← Standard basis

$B = B_y \circ T \circ B_y^{-1}$ ← composition of linear map from $\mathbb{F}^n \to \mathbb{F}^n$

← see why

$A = B_x \circ T \circ B_x^{-1}$

$$B = B_y \circ T \circ B_y^{-1} = B_y \circ B_x^{-1} \circ \underbrace{B_x \circ T \circ B_x^{-1}}_{A} \circ B_x \circ B_y^{-1}$$

$$B = B_y \circ B_x^{-1} \circ A \circ B_x \circ B_y^{-1}$$
$$= B_y \circ B_x^{-1} \circ A \circ (B_y \circ B_x^{-1})^{-1} = S A S^{-1}$$

$\mathbb{F}^n \xrightarrow{B_x^{-1}} X \xrightarrow{B_y} \mathbb{F}^n$

isomorphism ∴ invertible

$S \in M_{n\times n}(\mathbb{F})$

Note: $B = SAS^{-1}$ then they are called similar

Exercise: Let $T$ be a linear map from $X$ to $X$ and $A$ be a matrix representation of $T$ w.r.t a basis of $\{x_1, \dots x_n\}$. Let $B = SAS^{-1}$, $\exists S \in M_{n\times n}(\mathbb{F})$. Then find a basis $y = \{y_1, y_2, \dots y_n\}$ of $X$ s.t $B$ is the matric rep of $T$ w.r.t $y$.

done see down

now $\mathbb{F}^n \xrightarrow{B_x^{-1}} X \xrightarrow{B_y} \mathbb{F}^n$

$B_y \circ B_x^{-1} = S$  As $B_x(x_i) = e_i$

$$B_y \circ B_x^{-1}(\varepsilon_\ell) = B_y(x_\ell) = B_y\left(\sum_{i=1}^{n} A_{i\ell}\, y_i\right)$$
$$= \sum_{i=1}^{n} A_{i\ell}\, e_i$$

$$S = \begin{array}{c} \begin{bmatrix} & A_{1\ell} & \\ & A_{2\ell} & \\ & \vdots & \\ & A_{n\ell} & \end{bmatrix} \\ \ell^{th} \text{ column} \end{array}$$

**Determinants:**

let $D: \underbrace{\mathbb{F}^n \times \dots \times \mathbb{F}^n}_{n \text{ times}} \longrightarrow \mathbb{F}$ be a map such that it satisfies the below prop:

P1: $D(x_1, x_2, \dots x_n) = 0$ if $x_i = x_j$ for some $i \neq j$

P2: Multilinear:

$$D(x_1, x_2, \dots x_{i-1}, \alpha x + y, x_{i+1}, \dots x_n)$$

$$= \alpha D(x_1, x_2, \dots x_{i-1}, x, x_{i+1}, \dots x_n) + D(x_1, x_2, \dots x_{i-1}, y, x_{i+1}, \dots x_n)$$

All fixed but one is $\alpha x + y$ then with to that column it is linear (At any column) (Multilinearity)

P3: Normalisation:

$$D(e_1, e_2, \dots e_n) = 1$$

**Lemma**: Let $D, \mathbb{F}^n \times \dots \mathbb{F}^n \to \mathbb{F}$ be s.t it satisfies $P_1, P_2$ and $P_3$. Then

$$D(x_1, x_2, \dots x_n) = -D(y_1, y_2, \dots y_n)$$

where $y_i = \begin{cases} x_i, & \forall i \neq p, q \\ x_p, & i = q \\ x_q, & i = p \end{cases}$

**proof**: Set $D(x_p, x_q) = D(x_1, x_2 \dots x_{p-1}, x_p, x_{p+1}, \dots x_q, \dots x_n)$

to prove: $D(x_p, x_q) = -D(x_q, x_p)$

$$\begin{aligned}
D(x_p, x_q) &= D(x_p, x_q) + D(x_q, x_q) \\
&= D(x_p + x_q, x_q) \\
&= D(x_p + x_q, x_q) - D(x_p + x_q, x_p + x_q) \\
&= D(x_p + x_q, -x_p) \\
&= -D(x_p + x_q, x_p) \\
&= -D(x_q, x_p) - D(x_p, x_p) \\
&= -D(x_q, x_p)
\end{aligned}$$

**Lemma**: Let $D$ be as above, If $\{x_1, \dots x_n\}$ is a dependent set of vectors in $\mathbb{F}^n$ then $D(x_1, x_2, \dots x_n) = 0$.

**proof**:

Suppose $x_1 = \sum_{i=2}^{n} \alpha_i x_i$ then $D(x_1, x_2, \dots x_n) = D(\sum_{i=2}^{n} x_i \alpha_i, x_2, x_3, \dots x_n)$

$$= \sum_{i=2}^{n} \alpha_i D(x_i, x_2, \dots x_n)$$

$= 0$ By Property 1.

**Def^n**: (Signature of Permutation)

$$\sigma : (1, 2, \dots n) \longrightarrow (1, 2, \dots n)$$

$\sigma \in S_n$ permutation

define $P(x_1, x_2, \dots x_n) = \prod_{i<j} (x_i - x_j)$

$$P(x_{\sigma(1)}, \dots x_{\sigma(n)}) = \prod_{i<j} (x_{\sigma(i)} - x_{\sigma(j)})$$

$\text{sgn}(\sigma)$ is s.t

$$P(x_1, x_2, \dots x_n) = \text{sgn}(\sigma) P(x_{\sigma(1)}, \dots x_{\sigma(n)})$$

Now, $D(x_1, x_2, \dots x_n) = D(\sum_{i=1}^{n} a_{1i} e_i, x_2, \dots x_n)$

$$= \sum_{i=1}^{n} a_{1i} D(e_i, x_2, \dots x_n)$$

$$\vdots$$

$$= \sum a_{1f(1)} \cdots a_{nf(n)} D(e_{f(1)}, \dots e_{f(n)})$$

where the sum $f : \{1, \dots n\} \to \{1, \dots n\}$ is over all functions

see if

$$D(x_1, x_2) = D(a_{11} e_1 + a_{12} e_2, x_2)$$
$$= a_{11} D(e_1, x_2) + a_{12} D(e_2, x_2)$$
$$= a_{11} a_{21} D(e_1, e_1) + a_{11} a_{22} D(e_1, e_2)$$
$$+ a_{12} a_{21} D(e_2, e_1) + a_{12} a_{22} D(e_2, e_2)$$

$$= \sum a_{1f(1)} \dots a_{nf(n)} D(e_{f(1)} \dots e_{f(n)})$$

Reduces to

$$\rightsquigarrow = \sum_{\sigma \in S_n} a_{1\sigma(1)} \dots a_{n\sigma(n)} D(e_{\sigma(1)}, \dots e_{\sigma(n)})$$

$$= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

<u>Exercise</u>: Let $T$ be a linear map from $X$ to $X$ and $A$ be a matrix representation of $T$ w.r.t a basis of $\{x_1, \dots x_n\}$. Let $B = SAS^{-1}$, $\exists S \in M_{n \times n}(\mathbb{F})$. Then find a basis $y = \{y_1, y_2, \dots y_n\}$ of $X$ s.t $B$ is the matrix rep of $T$ w.r.t $y$.

Here as $B = SAS^{-1}$

and $A = B_x \circ T \circ B_x^{-1}$

where $B_x(x_i) = e_i$ ← standard basis

now

$$B = S(B_x \circ T \circ B_x^{-1}) S^{-1}$$
$$= (SB_x) T (SB_x)^{-1}$$

now for $SB_x = B_y$

then

$$SB_x(y_i) = e_i$$

as $SB_x$ will be a linear map from $X$ to $\mathbb{F}^n$ and will be invertible

we have

$$y_i = (SB_x)^{-1}(e_i)$$
$$y_i = B_x^{-1} S^{-1}(e_i)$$

$\therefore$ Basis will be $B_x^{-1} S^{-1}(e_i)$

where $A = B_x \circ T \circ B_x^{-1}$

$$A(e_i) = B_x \circ T \circ B_x^{-1}(e_i)$$

5th sept:

## $S_n$ and $\sigma$ notations:

$S_n$ - group of permutations on n-symbols

also called symmetric group

$$\sigma : \{1, \dots n\} \xrightarrow{\text{bijective}} \{1, \dots n\}$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \dots & \sigma(n) \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} \sigma(1) & \sigma(2) & \dots & \sigma(n) \\ 1 & 2 & & n \end{pmatrix}$$

$$\sigma \circ \sigma^{-1} (\sigma(1)) = \sigma(1)$$

$$\sigma \circ \sigma^{-1} (\sigma(r)) = \sigma(r)$$

$$\sigma \circ \sigma^{-1} (i) = \sigma \circ \sigma^{-1}(\sigma k) = \sigma(k) = i$$

$$i = \sigma(k)$$

$$\Rightarrow \sigma \circ \sigma^{-1}(i) = i$$

**Defn: cycle:**

Let $S_n$ be the permutation group of $\{x_1 \dots x_n\}$ then $(x_1, \dots x_k)$ is a cycle where

$$(x_1, x_2, \dots x_k)(x_i) = \begin{cases} x_i & \text{if } x_i \notin \{x_1 \dots x_k\} \\ x_{i+1} & \text{if } i \neq k \\ x_1 & \text{if } i = k \end{cases}$$

**Note:** $(x_1, x_2, \dots x_k) = (x_2, x_3, \dots x_k, x_1)$

**Defn:** length of cycle $(x_1, x_2, \dots x_k)$ - k-cycle (length)

2-cycle is called transposition.

Eg: $(x, y)$ transposition, just changes x and y position.

**product of k-cycles.**

We take an example:

Every $\sigma$ can be written as multiplication of product of cycles

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 1 & 5 \end{pmatrix}$$

Note $1 \to 3 \to 2 \to 4 \to 1 \to 3$

so $(1\,3\,2\,4)$ product of cycles

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$$

$1 \to 2 \to 5 \to 1 \to 2$ cycle 1

$3 \to 4 \to 3$ cycle 2

$(1\,2\,5)(3\,4)$

$= (3\,4)(1\,2\,5)$

**Note:** k-cycle

$(1\,2\,3) = (1\,3)(1\,2)$

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix}$$

similarly

$(x_1, x_2, \dots x_k) = (x_1, x_k)(x_1, x_{k-1}) \dots (x_1, x_2)$

**proof:** if $x_i \notin \{x_1, \dots x_k\}$

then

$(x_1, x_2, \dots x_k)(x_i) = x_i$

and same for $(x_1, x_k) \cdots (\cdot) x_i = x_i$

if $x_i \in \{x_1, \dots x_k\}$ and $x_i \neq x_k$

then $(x_1, \dots x_k)(x_i) = x_{i+1}$

and $(x_1, x_k)(x_1, x_{k-1}) \dots (x_1, x_2)(x_i)$

$$= \begin{cases} x_2 ; & i = 1 \\ x_{i+1} ; & i \neq 1 \end{cases}$$

This is not unique

$(1\,2\,3) = (1\,3)(1\,2)$

$= (4\,5)(4\,5)(1\,3)(1\,2)$

$\sigma$ as two same representations

similarly for $i = k$ we see both sides equal.

$\therefore$ every cycle of length $> 2$ can be converted into multiplication of cycles of size 2

Every permutation can be written as product of 2 cycles. they are both 2 cycles

**Theorm**: (Parity theorm) let $\sigma \in S_n$. let $\sigma = \tau_1 \dots \tau_n$ and let $\sigma = \gamma_1' \dots \tau_m'$ be factorisation of $\sigma$ as product of 2-cycles

Then $|n-m| = 2i$, $i \in \mathbb{N} \cup \{0\}$

**Def$^n$**: Let $\sigma \in S_n$ and $\sigma = \tau_1 \dots \tau_n$ be a factorisation of $\sigma$ as product of two cycles, Then

$\operatorname{sgn}(\sigma) = (-1)^n$ Because of Parity theorm this will be unique

**Note**: $\sigma = \tau_1 \tau_2 \dots \tau_n$

and as $(\tau_i)^{-1} = \tau_i$ we have

$$\sigma^{-1} = (\tau_1 \tau_2 \dots \tau_n)^{-1} = \tau_n \tau_{n-1} \dots \tau_1$$

$$\sigma \cdot \sigma^{-1} = \tau_1 \tau_2 \dots \tau_n \tau_n \tau_{n-1} \dots \tau_1$$
$$= \tau_1 \tau_1$$

determinants again:

$$D: \underbrace{\mathbb{F}^n \times \mathbb{F}^n \times \dots \mathbb{F}^n}_{n \text{ times}} \to \mathbb{F}$$

1. $D(a_1, \dots a_n) = 0$ if $a_i = a_j$, $i \neq j$
2. multilinear i.e
$$D(a_1, \dots \lambda a + b, \dots a_n) = \lambda D(a_1, \dots a, \dots a_n) + D(a_1, \dots b, \dots a_n)$$
$\therefore$ a linear map $\forall i = 1, 2, \dots n$

3. $D(e_1, \dots e_n) = 1$

**Note**: $D(a_1, \dots a_n)$

where
$$a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} = a_{1j} e_1 + a_{2j} e_2 + a_{3j} e_3 + \cdots$$
$$= \sum_{i=1}^{n} a_{ij} e_i$$

(last class we did the row method, this one we follow now)

then $D(a_1, a_2, \dots a_n)$

$$= D\left(\sum_{i=1}^{n} a_{i1} e_i, \sum_{i=1}^{n} a_{i2} e_i, \dots \sum_{i=1}^{n} a_{in} e_n\right)$$
$$= \sum_f a_{f(1)1} \cdots a_{f(n)n} D(e_{f(1)}, e_{f(2)}, \dots e_{f(n)})$$
$$= \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n} D(e_{\sigma(1)}, \dots e_{\sigma(n)})$$

**To prove**: $\sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n} D(e_{\sigma(1)}, \dots e_{\sigma(n)})$
$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

**proof**: $D(e_{\sigma(1)}, \dots e_{\sigma(n)})$ $\quad 1 \to \sigma(1)$
$= D(\sigma(e_1, \dots e_n))$ $\quad e_1 \to e_{\sigma(1)}$

$$D(e_1, \dots e_n) = D(\sigma^{-1} \circ \sigma(e_1, \dots e_n))$$
$$= D(\sigma^{-1} \circ (e_{\sigma(1)}, \dots e_{\sigma(n)}))$$

Because of the alternative prop now $\sigma = \tau_1 \dots \tau_k$ that we proved already $D(a, b) = -D(b, a)$

then
$$= (-1)^k D(e_{\sigma(1)}, \dots e_{\sigma(n)})$$
$$\Rightarrow \operatorname{sgn}(\sigma) = (-1)^k$$
$\therefore$ $D(e_{\sigma(1)}, \dots e_{\sigma(n)}) = \operatorname{sgn}(\sigma)$ as $D(e_1, \dots e_n) = 1$

Defn: Let $A$ be a $n \times n$ matrix over $\mathbb{F}$. If $A = (a_1, \ldots a_n)$, where $a_i \in \mathbb{F}^n$ then $\det A = D(a_1, \ldots a_n)$

Note: $\det(AB) = \det(A)\det(B)$

$9^{th}$ sept: recap:

$A = (a_1, \dots a_n)$

$\det(A) = D(a_1, \dots a_n)$

$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}$

where $a_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}, \forall i = 1, 2, \dots n$

$\begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & \vdots \\ \vdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$

**Theorem:** for $A, B \in M_{n \times n}(\mathbb{F})$,
$\det(AB) = \det(A)\det(B)$

**proof:**

we know $\det(AB) = D(AB(e_1), AB(e_2), \dots AB(e_n))$
where $\{e_1, e_2, \dots e_n\}$ are standard basis of $\mathbb{F}^n$.

$\Rightarrow \det(AB) = D(AB_{*1}, AB_{*2}, \dots AB_{*n})$

case I:

$\det(A) \neq 0$ $\quad C: \mathbb{F}^n \times \mathbb{F}^n \dots \mathbb{F}^n \to \mathbb{F}$

define a map $C(x_1, x_2, \dots x_n) = \dfrac{D(A(x_1), A(x_2), \dots A(x_n))}{\det(A)}$

Note: $x_i \in \mathbb{F}^n$
(Not necessarily $B_{*i}$)

now, we want to see if $C$ satisfies $P_1, P_2, P_3$

① let $x_i = x_j$ for $i \neq j$
then $C(x_1, x_2, \dots x_n) = \dfrac{D(A(x_1), \dots A(x), \dots A(x), \dots A(x_n))}{\det(A)}$

$= \dfrac{0}{\det(A)} = 0$

as $A(x_i) = A(x_j)$

② let $x_2, \dots x_n \in \mathbb{F}^n$ be fixed then
$C(\alpha x + \beta y, x_2, \dots x_n) = \dfrac{D(A(\alpha x + \beta y), A(x_2) \dots A(x_n))}{\det(A)}$

$\alpha C(x \dots) + \beta C(y \dots) \quad = \dfrac{\alpha D(A(x), A(x_2) \dots) + \beta D(A(y), A(x_2) \dots)}{\det(A)}$

$\therefore$ linear column wise
(multilinear)

③ $C(e_1, e_2, \dots e_n) = \dfrac{D(A_{*1}, A_{*2}, \dots A_{*n})}{\det(A)}$

$= 1$ as $\det(A)$
$= D(A_{*1}, \dots A_{*n})$

By the uniqueness of $D$, we have

$\det(B) = \dfrac{\det(AB)}{\det(A)}$

$\Rightarrow \det(AB) = \det(A)\det(B)$

case II) $\det(A) = 0$

Consider $A(t) = A + tI$ for $t \in [0,1]$

$f: t \in [0,1] \to \det(A(t))$ is a monic polynomial of degree $n$

$\therefore$ atmost $n$ zeroes (finitely many $t$'s)

(as $\left|\begin{bmatrix} a+t & b \\ c & d+t \end{bmatrix}\right| \to \deg 2$)

$\therefore$ from 0 to the point where $\det(A(t_0)) = 0$ we can find many points where $\det(A(t)) \neq 0$

$t \in (0, t_0)$

then by case I, $\det(A(t)B) = \det(A(t))\det(B)$, $\forall\, t \in (0, t_0)$

$$\begin{bmatrix} a_{11}+t & \cdot & \cdots \\ \vdots & a_{22}+t & \cdots \\ \cdots & \cdots & a_{nn}+t \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & & \\ \vdots & & \end{bmatrix} \to \det(A(t)B)$$

can have atmost $t$.

Now $\det(A(t)B)$ is a polynomial of degree atmost $n$ and $\det(A(t))$ is a polynomial of degree $n$ (monic)

Therefore by taking limit as $t \to 0$ in $\det(A(t))\det(B)$ we get $\det(A)\det(B) = \det(AB)$

**Proposition**: Let $A \in M_{n\times n}(\mathbb{F})$. Then $A$ is invertible iff $\det(A) \neq 0$

**proof**: ($\Rightarrow$) If $A$ is invertible then

$\det(I) = \det(A)\det(A^{-1})$

$\Rightarrow 1 = \det(A)\det(A^{-1})$

$\Rightarrow \det(A) = \dfrac{1}{\det(A^{-1})}$ $\therefore \det(A) \neq 0$.

($\Leftarrow$) $0 \neq \det(A) = D(A_{*1}, A_{*2}, \ldots A_{*n})$

this implies $\{A_{*1}, A_{*2}, \ldots A_{*n}\}$ are linearly independent

($\because$ if lin dep then $D = 0$) (Here colums are lin ind, rank $= n$, range $= \mathbb{F}^n$ $\therefore R_T = \mathbb{F}^n$)

As $\{A_{*1}, \ldots A_{*n}\}$

Note, $\{A_{*1}, \ldots A_{*n}\}$ is lin ind follows from a lemma already proved.

$\Rightarrow$ rank $= n$

$\Rightarrow R_A = \mathbb{F}^n$

$\Rightarrow$ nullity $= 0$

$\therefore A: \mathbb{F}^n \to \mathbb{F}^n$ is one-one and onto

$\therefore A$ is isomorphism

$\Rightarrow A$ is invertible

Exercise: Show that similar matrices have same determinant. ← done

Theorm : for $A \in M_{n\times n}(\mathbb{F})$, $\det(A) = \det(A^T)$

proof : If $A = (a_1, a_2, \ldots a_n)$, then $\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a'_{1\sigma(1)} \cdots\cdots a'_{n\sigma(n)}$$

where $a^T = (a_1', a_2', \ldots a_n')$

Note : $a'_{i\sigma(i)} = a'_{\sigma^{-1}(j)j}$

so $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a'_{1\sigma(1)} \cdots\cdots a'_{n\sigma(n)}$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma^{-1}) a'_{\sigma^{-1}(1)(1)} \cdots a'_{\sigma^{-1}(n)n}$$

as $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$

$$= \sum_{\sigma^{-1} \in S_n} \operatorname{sgn}(\sigma^{-1}) a'_{\sigma^{-1}(1)(1)} \cdots a'_{\sigma^{-1}(n)(n)}$$

(as $S_n$ is a group, $\sigma \in S_n \Rightarrow \sigma^{-1} \in S_n$)

$$= \det(A^T)$$

$\therefore \det(A) = \det(A^T)$

$\det(A) = \sum_i (-1)^{1+i} a_{i1} A_{i1}$ minor

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ \vdots & & A & & \\ 0 & & & & \end{bmatrix} = B$$

then $\det(A) = \det(B)$

as $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots\cdots$

here $a_{\sigma(1)1} = a_{11}$

else it is zero

$$\therefore \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \underbrace{a_{\sigma(2)2} \cdots\cdots}_{\text{this is } \det(A)}$$

$$= \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) a_{\sigma(2)2} \cdots\cdots$$

$\det(B) = \det(A)$

Exercise : Show that similar matrices have same determinant.

as $\exists S$ s.t

$B = SAS^{-1}$

$\det(B) = \det(A)\det(S)\det(S^{-1})$

$= \det(A)\det(I)$

$\det(B) = \det(A)$

12th sept -

$\det(A) = \sum_i (-1)^{i+j} (a_{ij}) A_{ij}$

lemma: $D(a_1, a_2, \dots a_{i-1}, e_j, a_{i+1}, \dots a_n) = D(a_1 - a_{1j}e_j, a_2 - a_{2j}e_j, \dots \dots a_{i-1} - a_{(i-1)j}e_j, e_j, \dots\dots a_n - a_{nj}e_j)$

$\forall a_i \in \mathbb{F}^n$, $e_j = (0,0,\dots,1,0,0..)$ ← $j^{th}$ place

$a_1 - a_{1j}e_j = \begin{pmatrix} a_{11} \\ \vdots \\ 0 \\ \vdots \\ a_{1n} \end{pmatrix}$ ← $j^{th}$ place $\leftarrow a_1 = \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{pmatrix}$

$$\det \begin{bmatrix} a_{11} & \cdots & \\ a_{21} & & \\ \vdots & & \\ a_{n1} & & a_{nn} \end{bmatrix} = \det \begin{bmatrix} a_{11} & & 0 & a_1 \\ \vdots & & 0 & \\ 0 & 0 \cdots 0 & 1 & \cdots 0 \\ a_{n1} & & 0 & \cdots a_{nn} \end{bmatrix}$$

proof: $D(a_1, a_2 \dots a_n) = D(a_1 + \lambda a_2, a_2, a_3, \dots a_n)$
as $D(\lambda a_2, a_2, \dots a_n) = 0$

$\Rightarrow D(a_1, a_2, \dots e_j, \dots a_n) = D(a_1 + \lambda_1 e_j, a_2 + \lambda_2 e_j, \dots a_n)$

$\lambda_i = -a_{ij}$ other than $i = j$

$\Rightarrow D(a_1, a_2, \dots e_j, \dots a_n) = D(a_1 - a_{1j}e_j, a_2 - a_{2j}e_j, \dots e_j, \dots a_n - a_{nj}e_j)$

lemma: $\det \begin{pmatrix} 1 & x & x \cdots x \\ 0 & & \\ \vdots & & A \\ 0 & & \end{pmatrix}_{n \times n} = \det(A)$

proof: $\det \begin{pmatrix} 1 & x & x \cdots x \\ 0 & & \\ \vdots & A & \\ 0 & & \end{pmatrix}_{n \times n} = \det \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix}_{n \times n}$

from the previous lemma

$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & \sigma(2) & & \sigma(n) \end{pmatrix}$

$sgn(\sigma) = sgn(\sigma')$

$\sigma' = \begin{pmatrix} 2 & 3 & \cdots n \\ \sigma(2) & \sigma(3) & \sigma(n) \end{pmatrix}$

Define $c : \mathbb{F}^{n-1} \times \mathbb{F}^{n-1} \times \cdots \mathbb{F}^{n-1} \to \mathbb{F}$

$c(a_1, a_2, \dots a_{n-1}) = D(e_1, \hat{a}_1, \hat{a}_2, \dots \hat{a}_{n-1})$

where $\hat{a}_i = \begin{pmatrix} 0 \\ a_{1i} \\ a_{2i} \\ \vdots \\ a_{n-1 i} \end{pmatrix}$, $\forall i = \{1, 2, \dots n-1\}$

① $c(a_1, a_1, a_3, \dots a_{n-1}) = D(e_1, \hat{a}_1, \hat{a}_1, \dots \hat{a}_{n-1}) = 0$

② $c(a_1, a_2, \dots \lambda a + b, \dots a_{n-1}) = D(e_1, \hat{a}_1, \dots \lambda \hat{a} + \hat{b}, \dots \hat{a}_{n-1})$

$= \lambda D(e_1, \hat{a}_1, \dots \hat{a}, \dots \hat{a}_{n-1}) + D(e_1, \hat{a}_1, \dots \hat{b}, \dots \hat{a}_{n-1})$

$= \lambda c(a_1, \dots a, \dots a_{n-1}) + c(a_1, \dots b, a_{n-1})$

③ $c(e_1, e_2, \dots e_{n-1}) = D(e_1, \hat{e}_1, \hat{e}_2, \dots \hat{e}_{n-1})$
$= D(e_1, e_2, e_3, \dots e_n)$
$= 1$

as $c$ satisfies all these properties,

$c(a_1, a_2, \dots \dots a_{n-1}) = \det(A)$

$\therefore \det \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix} = \det(A)$

**Lemma**: If $j^{th}$ column of a matrix $A$ is $e_i$, then $\det(A) = (-1)^{i+j} A_{ij}$, where $A_{ij}$ is the matrix obtained from $A$ by striking $i^{th}$ row and $j^{th}$ column.

$$\det \begin{array}{c} \overbrace{\hphantom{a_{11}\ a_{12}\cdots 0}}^{i^{th}\text{ column}} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & 0 & \cdots \\ \vdots & & & 0 & \\ a_{j1} & \cdots & \cdots & 1 & \cdots \\ & & & 0 & a_{nn} \end{pmatrix}_{n\times n} \end{array} = (-1)^{i-1} \det \begin{pmatrix} 0 & a_{11} & & \vdots \\ 0 & \vdots & & \vdots \\ \vdots & & & \\ 1 & 0 & 0 \cdots & 0 \\ 0 & \vdots & & \\ 0 & & & \end{pmatrix}_{n\times n}$$

we swap $i-1$ times

$$= (-1)^{i-1} \det \begin{pmatrix} 0 & & & \\ 0 & & & \\ \vdots & & & \\ 1 & 0 & \cdots & 0 \\ 0 & & & \end{pmatrix}^T_{n\times n}$$

$$= (-1)^{i-1+j-1} \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_{ij}^T & \\ 0 & & & \end{pmatrix}_{n\times n}$$

$$= (-1)^{i+j} \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_{ij} & \\ 0 & & & \end{pmatrix}_{n\times n}$$

**proof**: We define $C : \mathbb{F}^{n-1} \times \mathbb{F}^{n-1} \times \cdots \mathbb{F}^{n-1} \to \mathbb{F}$

$$\det \begin{pmatrix} & & & 0 & & \\ & & & 0 & & \\ \times & \times & \times & \times\!\!\overset{\vdots}{1} & \times & \times \\ & & & 0 & & \end{pmatrix}_{n\times n} = \det \begin{pmatrix} \vdots & & & 0 & \vdots \\ & & & 0 & \\ & & & 0 & \\ 0 & 0 & 0\ 0 & 1 & 0\ 0 \\ \vdots & & & 0 & \vdots \end{pmatrix}_{n\times n} = C(a_1, a_2, \ldots a_{n-1})\,(-1)^{i+j}$$

$C : \mathbb{F}^{n-1} \times \mathbb{F}^{n-1} \times \cdots \times \mathbb{F}^{n-1} \longrightarrow \mathbb{F}$ by

$$C(a_1, a_2, \ldots a_{n-1}) = (-1)^{i+j} D(\hat{a_1}, \hat{a_2}, \ldots \hat{a}_{j-1}, e_i \ldots \hat{a}_{n-1})$$

where $\hat{a}_k = \begin{pmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{i-1\,r} \\ 0 \\ a_{n-1\,r} \end{pmatrix}$ check that $C$ has all these properties ← done down

**Theorem**: Let $A \in M_{n\times n}(\mathbb{F})$, then $\det(A) = \sum (-1)^{i+j} a_{ij} \det(A_{ij})$ where, $A = (a_{ij})_{i,j=1}^n$ and $A_{ij} \in M_{(n-1)\times(n-1)}(\mathbb{F})$ obtained by taking out $i^{th}$ row and $j^{th}$ column of $A$.

**proof**: wlog $j = 1$

$$\begin{aligned} \det(A) &= D(a_1, \ldots a_n) \\ &= D(\textstyle\sum a_{i1} e_i, \ldots a_n) \\ &= \textstyle\sum a_{i1} D(e_i, \ldots a_n) \\ &= \textstyle\sum a_{i1} (-1)^{i+1} D(A_{i1}) \end{aligned}$$

Inverse calculation of a determinant ← done down

**Defn**: $A \in M_{n\times n}(\mathbb{F})$, then $tr(A) = \sum_{i=1}^{n} a_{ii}$

**Theorm**: $tr: M_{n\times n}(\mathbb{F}) \longrightarrow \mathbb{F}$ is a linear fuction. Moreover $tr(AB) = tr(BA)$

**proof**: if $tr(\alpha A + \beta B) = \alpha\, tr(A) + \beta\, tr(B)$ $\forall A, B \in M_{n\times n}(\mathbb{F})$, $\alpha, \beta \in \mathbb{F}$

$$\begin{aligned} tr(\alpha A + \beta B) &= \sum (\alpha a_{ii} + \beta b_{ii}) \\ &= \alpha \sum a_{ii} + \beta \sum b_{ii} \\ &= \alpha\, tr(A) + \beta\, tr(B) \end{aligned}$$

$$\begin{aligned} tr(AB) = \sum (AB)_{ii} &= \sum_{i=1}^{n} \Big(\sum_{k=1}^{n} a_{ik} b_{ki}\Big) \\ &= \sum_{k=1}^{n} \Big(\sum_{i=1}^{n} b_{ki} a_{ik}\Big) \\ &= \sum_{k=1}^{n} (BA)_{kk} \\ &= \sum_{i=1}^{n} (BA)_{ii} = tr(BA) \end{aligned}$$

**propn**: If $A, B$ are similar matrices then $tr(A) = tr(B)$

**proof**: $A = SBS^{-1}$

$$\begin{aligned} tr(A) &= tr(SBS^{-1}) \\ &= tr(S^{-1}SB) \\ &= tr(B) \end{aligned}$$

show $\det \begin{pmatrix} \vdots & & & 0 & \vdots \\ & & & 0 & \\ 0\ 0 & 0\ 0 & & 1 & 0\ 0 \\ \vdots & & & 0 & \vdots \end{pmatrix}_{n\times n} = c(a_1, a_2, \dots a_{n-1})\, (-1)^{i+j}$, $c$ is a determint

$c: \underbrace{\mathbb{F}^{n-1} \times \mathbb{F}^{n-1} \times \cdots \mathbb{F}^{n-1}}_{n-1 \text{ times}} \longrightarrow \mathbb{F}$

$$\begin{aligned} c(a_1, a_2, \dots a_{n-1}) &= D(\hat{a}_1, \hat{a}_2, \dots \hat{a}_{i-1}, e_j, \hat{a}_{i+1}, \dots \hat{a}_n) \\ &= (-1)^{i+j} D(e_j, \hat{a}_1, \hat{a}_2, \dots \hat{a}_{i-1}) \\ \det(A_{ij}) &= (-1)^{i+j} \det(A) \end{aligned}$$

$\Rightarrow \det(A) = (-1)^{i+j} \det(A_{ij})$

Inverse calulation of a determinant

$(-1)^{i+j} A_{ij} = \text{cofactor} = C_{ij}(A)$

$\operatorname{adj} A = [C_{ij}(A)]^T$

$A(\operatorname{adj} A) = \det(A)\, I$

and $A(\operatorname{adj} A) = [a_{ij}][C_{ij}(A)]^T$

$\Rightarrow A^{-1} = \dfrac{\operatorname{adj} A}{\det(A)}$

also

$$\begin{bmatrix} 1 & 2 & 6 & 1 & 2 \\ 2 & 3 & 4 & 2 & 3 \\ 3 & 4 & 2 & 3 & 4 \\ 1 & 2 & 6 & 1 & 2 \\ 2 & 3 & 4 & 2 & 3 \end{bmatrix}^T \times \frac{1}{\det A} = A^{-1}$$

23rd sept -

**Defn**: The eigenvalue of $A \in M_{n\times n}(\mathbb{C})$ is $\lambda \in \mathbb{C}$ s.t $Ax = \lambda x$ for some non-zero vector $x \in \mathbb{C}^n$. In such a case the vector $x$ is called the eigenvector corr. to the eigenvalue $\lambda$.

Given $A \in M_{n\times n}(\mathbb{C})$, we consider a non-zero vector $x \in \mathbb{C}^n$. clearly $\forall u$, $Au$, $A^2u, A^3u, \ldots A^n u \in \mathbb{C}^n$

we know $\{u, Au, \ldots A^n u\}$ is linearly dependent.

(as $\dim \mathbb{C}^n = n$, and $u, Au, \ldots A^n u$ is $n+1$)

$\exists\, c_i \in \mathbb{C}$, not zero s.t

$$\sum_{i=0}^{n} c_i A^i u = 0 \qquad c_i \neq 0 \text{ ——— } ①$$

let $p(t) = \sum c_i t^i$ for $t \in \mathbb{C}$

our polynomial (degree $n$, $n$ roots say $\lambda_1, \lambda_2, \ldots \lambda_n$)

$$p(t) = c_n \prod_{i=1}^{n} (t - \lambda_i) \text{ for some } \lambda_i \in \mathbb{C}$$

now,

$$p(A)u = \underbrace{\left(\sum_{i=0}^{n} c_i A^i\right) u = 0}_{\text{from } ①}$$

$\Leftrightarrow \prod_{i=1}^{n} (A - \lambda_i I) u = 0$ $(u \neq 0)$ $(\ker \neq \{0\})$

$\Leftrightarrow \prod_{i=1}^{n} (A - \lambda_i I)$ is non invertible ($\because$ ker contains a non-zero $\lambda_i$)

$\Leftrightarrow \det(A - \lambda_i I) = 0$ for some $i$

$\Leftrightarrow (A - \lambda_i I)y = 0$ for some $0 \neq y \in \mathbb{C}^n$

$\Leftrightarrow Ay = \lambda_i y$ for some $0 \neq y \in \mathbb{C}^n$

**Result**: $\lambda$ is an eigen-value of $A \Leftrightarrow \det(A - \lambda I) = 0$

**Defn**: The characteristic polynomial of $A$ is:

$$\lambda \longmapsto \det(A - \lambda I)$$

$\lambda$ is an eigenvalue of $A$ $\Leftrightarrow$ $\lambda$ is an root for the char. polynomial

**Note** for $A \in M_{n\times n}(\mathbb{C})$, $u_0$

where $Au_0 = \lambda' u_0$

$$A.Au_0 = A\lambda' u_0$$
$$= Au_0 \lambda'$$
$$= \lambda' u_0 \lambda'$$
$$A^2 u_0 = \lambda'^2 u_0$$

or $A^n u_0 = \lambda^n u_0$

$\forall n = 0, 1, 2, \ldots n$

**Defn**: Markov matrix:

① Entries of a markov matrix is always positive

② sum of entries in each column is 1

**Note**: If $\lambda$ is an eigen value, then $\lambda$ is also an eigenvalue of $A^T$ as

$$(A - \lambda I)^T = A^T - \lambda I^T$$
$$= A^T - \lambda I$$

as $\det(A - \lambda I)^T = \det(A - \lambda I) = \det(A^T - \lambda I)$

$A \to$ markov matrix, then sum of columns =

$$A^T = \begin{bmatrix} \text{———} \\ \text{———} \\ \text{———} \\ \text{———} \end{bmatrix} \to \text{sum of rows} = 1$$

$$A^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \text{———} \\ \text{———} \\ \text{———} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

so $x = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is a eigenvector to $A^T$

$\Rightarrow \lambda = 1$ is eigenvalue to $A^T$

$\Rightarrow \lambda = 1$ is eigenvalue to $A$

## Properties of markov matrices : ($A \in M_{n \times n}(\mathbb{C})$) (general case)

① All the remaining eigenvalues of $A$ have modulus strictly less than 1.

② $A$ has $n$ no. of linearly idependent eigenvectors. $\{x_1, x_2, \dots x_n\}$

where

$A x_i = \lambda_i x_i$

where $\lambda_1 = 1$

so $A x_1 = x_1$

Now $0 \neq u_0 \in \mathbb{C}^n$

by hypothesis,

$u_0 = \sum_1^n c_i x_i$ for some $c_i \in \mathbb{C}$

$$A u_0 = \sum_{i=1}^n c_i \lambda_i x_i$$

$$A^m u_0 = \sum_{i=1}^n c_i \lambda_i^m x_i$$

$$= c_1 x_1 + \sum_{i=2}^n c_i \lambda_i^m x_i$$

as $|\lambda_i| < 1$

at steady state

$$A^m u_0 = c_1 x_1$$

## Fibonacci numbers -

$0, 1, 1, 2, 3, 5, 8, \dots$

$$\left. \begin{aligned} F_{n+2} &= F_{n+1} + F_n \\ F_{n+1} &= F_{n+1} \end{aligned} \right\} \to V_{n+2} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} + F_n \\ F_{n+1} + 0 F_n \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}}_{V_{n+1}}$$

so $V_{n+2} = A V_{n+1}$

$V_{n+1} = A V_n$

$\vdots$

$$V_{n+1} = A^n V_1$$

$$= A^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = \left| \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \right| = (1-\lambda)(-\lambda) - (1) = 0$$

$$-\lambda + \lambda^2 - 1 = 0$$
$$\lambda^2 - \lambda - 1 = 0$$
$$\lambda = \frac{1 \pm \sqrt{1+4}}{2}$$
$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

or eigenvalues of $A = \lambda_1, \lambda_2$
$$= \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

now $Ax_1 = \lambda_1 x_1$

for $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x_1 = \left(\frac{1+\sqrt{5}}{2}\right) x_1$

$$x + y = \left(\frac{1+\sqrt{5}}{2}\right) x$$
$$x = \left(\frac{1+\sqrt{5}}{2}\right) y$$
$$x_1 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$$
$$x_2 = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underset{\downarrow \atop c_1}{\frac{1}{\sqrt{5}}} \underset{\downarrow \atop x_1}{\begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}} + \underset{\downarrow \atop c_2}{\frac{-1}{\sqrt{5}}} \underset{\downarrow \atop x_2}{\begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}}$$

$$v_1 = c_1 x_1 + c_2 x_2$$
$$v_{n+1} = A^n v_1$$
$$= A^n c_1 x_1 + A^n c_2 x_2$$
$$= c_1 \lambda_1^n x_1 + c_2 \lambda_2^n x_2$$
$$= c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n x_1 + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n x_2$$

then
$$F_{n+1} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \left(\frac{1+\sqrt{5}}{2}\right) - \frac{1}{\sqrt{5}}\underbrace{\left(\frac{1-\sqrt{5}}{2}\right)^n \left(\frac{1-\sqrt{5}}{2}\right)}_{< 1/2}$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \underbrace{\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}_{< 1/2}$$

$=$ nearest integer of $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$

(Keep the equation in mind)

26th Sept -

**Theorm:** let $A \in M_{n\times n}(\mathbb{C})$. The eigenvectors corresponding to different eigenvalues are linearly independent.

**proof:** let $v_i$ be an eigenvector corresponding to an eigenvalue $\lambda_i$, $i=1,2,..m$ and they are all different. ($\lambda_i \neq \lambda_j$) $\forall\, i\neq j$

Suppose that $\sum_{i=1}^{m} c_i v_i = 0$ and $c_i \neq 0$, $\forall\, i=1,2,\dots m$

$$\left(\underline{\text{Note}}: f(t) = \sum_{n=0}^{\sigma} a_n t^n \leftarrow \text{polynomial} \quad f(A) = \sum_{n=0}^{\sigma} a_n A^n \leftarrow A^0 = I\right)$$

now, $0 = f(A)\left(\sum_{i=1}^{m} c_i v_i\right) = \sum_{i=1}^{m} c_i f(A) v_i$

$$= \sum_{i=1}^{m} c_i f(\lambda_i) v_i$$

$$0 = \sum_{i=1}^{m} c_i f(\lambda_i) v_i$$

$$0 = \sum_{i=1}^{m} c_i f(\lambda_i) v_i$$ for any polynomial we choose $\forall\, f \in \underbrace{\mathbb{C}[z]}_{\text{polynomial}}$

in one variable, complex coeficients

taking a polynomial $f_i$ s.t

(lagrage polynomial) $f_i(\lambda_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & 0 \text{ otherwise} \end{cases}$

$$\left(\text{example}: f_i = \frac{\prod_{j\neq i}(t-\lambda_j)}{\prod_{j\neq i}(\lambda_i - \lambda_j)}\right)$$

Now, $f_i(\lambda_i) = 1$ else 0

so

putting $f = f_i$ we get

$$0 = \sum_{i=1}^{m} c_i f_i(\lambda_i) v_i$$

$$0 = c_i \underbrace{f_i(\lambda_i)}_{1} v_i + 0 + 0 \cdots 0$$

$$0 = c_i v_i$$

as $c_i \neq 0$

$\Rightarrow v_i = 0$

i.e the eigenvector is 0, which is a contradiction.

so, if $\sum_{i=1}^{m} c_i v_i = 0$ then $c_1 = c_2 = \dots c_m = 0$

**Theorm:** If $A \in M_{n\times n}(\mathbb{C})$ has distinct eigenvalues then there is a basis of $\mathbb{C}^n$ consist of eigenvectors of $A$.

**proof:** $A \in M_{n\times n}(\mathbb{C})$, $Ac_i = \lambda_i c_i$ $\forall\, i=1,2,\dots n$ (This is a special case)

$\Leftrightarrow$ $\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$ or $A$ is a is diagonal with diagonal enteries $\lambda_i$'s

(we have to show)

**defn**: **Diagonizable** - If a matrix $A \in M_{n\times n}(\mathbb{C})$ is diagonalizable if $\exists$ an invertible matrix $P \in M_{n\times n}(\mathbb{C})$ s.t $P^{-1}AP$ is a diagonal matrix.

Now, $P^{-1}AP$ is diagonal $\Leftrightarrow (P^{-1}AP)(e_i) = \lambda_i(e_i), \forall i = 1,2,\ldots n$

$\Leftrightarrow A P(e_i) = P\lambda_i(e_i)$

$\Leftrightarrow AP(e_i) = \lambda_i P_i$

$\Leftrightarrow A P_i = \lambda_i P_i$

(This is a general case)

Note $\{P(e_i) \mid i = 1,2\ldots n\}$ forms a basis consisting of eigenvectors of A.

**Note**: A is diagonizable $\Leftrightarrow$ A has n-distinct eigenvalues.

**characteristic polynomial of A**: $X_A(t) = \det(tI - A)$

**Note**: roots of $X_A$ are eigenvalues of A.

**Theorem**: If $a_1, a_2, \ldots a_n$ are eigenvalues of $A \in M_{n\times n}(\mathbb{C})$, then

$tr(A) = \sum_{i=1}^{n} a_i$

$\det(A) = \prod_{i=1}^{n} a_i$

and $X_A(t) = t^n - tr(A)t^{n-1} + \cdots (-1)^n \det A$

(sum of roots and product of roots approach)

**proof**: $X_A(t) = \prod_{i=1}^{n}(t - a_i)$

$= \det(tI - A)$

$$= \det \begin{bmatrix} t-a_{11} & \cdot & \cdot & \cdots & -a_{1n} \\ -a_{21} & t-a_{22} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ -a_{n1} & \cdots & \cdots & \cdots & t-a_{nn} \end{bmatrix}$$

the coefficients of $t^n, t^{n-1}$ appears from $\prod_{i=1}^{n}(t - a_{ii})$

$\Rightarrow$ The co-efficients of $t^n$ is one and $t^{n-1}$ is $-(\sum_{i=1}^{n} a_{ii})$
$= -tr(A)$

(Note, $\det(A) = \prod a_i$ follows from $t = 0$)

thus $X_A(t) = t^n - tr(A)t^{n-1} + \cdots (-1)^n \det(A)$

since roots of the char. polynomials are the eigenvalues of A,

$$X_A(t) = \prod_{i=1}^{n}(t - a_i) = t^n - (a_1 + \cdots a_n)t^{n-1} + \cdots + (-1)^n \prod_{i=1}^{n} a_i$$

By comparing coefficients, $tr(A) = \sum_{i=1}^{n} a_i$

$\det(A) = \prod_{i=1}^{n} a_i$

**Theorem:** (Spectral mapping theorem) for any polynomial $P \in \mathbb{C}[z]$ and $A \in M_{n\times n}(\mathbb{C})$, then $P(\sigma(A)) = \sigma(P(A))$

($\sigma(A)$ = spectrum of $A$ = set of all eigenvalues of $A$)

**Proof:** It is trivial to see that if $\lambda$ is an eigenvalue of $A$, then $P(\lambda)$ is an eigenvalue of $P(A)$.

$$\Rightarrow P(\sigma(A)) \subseteq \sigma(P(A))$$

Now, take $b \in \sigma(P(A))$ and consider

$$p(t) - b = c\prod_{i=1}^{n}(t - a_i)$$

(assume $P(t)$ is not const as const case is trivial)

as $b \in \sigma(P(A))$
$P(A) - bI$ is non invertible

$$P(A) - bI = c\prod_{i=1}^{n}(A - a_i I)$$

as product is non-invertible $\exists$ atleast one term non-invertible.

since $P(A) - bI$ is non-invertible,
$(A - a_i I)$ is non-invertible for some $1 \leq i \leq m$
therefore $a_i$ is an eigenvalue
or $a_i \in \sigma(A)$ and

$P(t) - b = c\prod_{i=1}^{m}(t - a_i)$ vanishes at $a_i$

$\Rightarrow P(a_i) = b$

or $b \in \sigma(P(A))$
then $b \in P(\sigma(A))$
as $a_i \in \sigma(A)$
and $P(a_i) = b$

$$\therefore \sigma(P(A)) \subseteq P(\sigma(A))$$

$$\therefore \sigma(P(A)) = P(\sigma(A))$$

**Note:** for $P = \chi_A(t)$

$$\sigma(\chi_A(A)) = \underbrace{\chi_A(\sigma(A))}_{\chi_A \text{ has roots as eigenvalues of } A} = \{0\}$$

30th sept:

$T \in M_{n\times n}(\mathbb{C})$

# Basis of $M_{n\times n}(\mathbb{C}) = n^2$

(for $M_{n\times m}(\mathbb{C})$ should be dim $= mn$)

$\underbrace{I, T, T^2, T^3, \dots T^{n^2}}$ has $n^2+1$ elements, and dim of $M_{n\times n}(\mathbb{C}) = n^2$,

$\{I, T, T^2, \dots T^{n^2}\}$ is lin dep.

$\sum_{i=0}^{n^2} \alpha_i T^i = 0$ for some $\underline{\alpha_i \neq 0}$.

$\Leftrightarrow P(T) = 0$, where $P(t) = \sum_{i=0}^{n^2} \alpha_i t^i$

Note: $P(T) = \sum_{i=0}^{n^2} \alpha_i T^i$

so $\exists T \in M_{n\times n}(\mathbb{C})$
s.t $P(T) = 0$

$\therefore$ given any square matrix, $\exists$ a non-zero polynomial which annihilates the matrix.

($\Upsilon = \{ p \in \mathbb{C}[z] \mid p(T) = 0 \}$ is an ideal because if $p \in \Upsilon$ and $q \in \mathbb{C}[z]$ then $pq(T) = p(T)q(T)$
$= 0 \cdot q(T)$
$= 0$

$\Rightarrow pq \in \Upsilon$)

Defn $I$ is a principle ideal if
$I = \langle p \rangle$ for some $p \in \mathbb{C}[z]$

Note: $p$ is unique monic polynomial
(If $p$ is smallest degree $p$ in $I$, as $\deg p \leq \deg x \; \forall x \in I$)

($q \in I$, $q = \underset{\in I}{p}\underset{\in I}{f} + h$ $\deg(h) < \deg(p)$
$\deg(h) = 0$

$\therefore$ every $q$ as form of $g = pf$
$\Rightarrow g \in \langle p \rangle$)

such $p$ is called a minimal polynomial of $T$.

Defn: $p$ is a minimal polynomial of $T$ if

1. $P(T) = 0$
2. if $Q(T) = 0$ then $q(x) = P(x)r(x)$ for some $r(x) \in \mathbb{C}[X]$
or $q \in \langle P \rangle$
or $p \mid q$

Note: How do we find minimal polynomial of $T$:

we know that $P'(T) = \sum \alpha_i T^i$
so $\deg(P) \leq n^2$

Theorem : The roots of minimal polynomial of T and charatristic polynomial of a matrix (T) are same.

roots of $\chi_T$ = roots of P (minimal polynomial)

proof : Enough to show that root of P $\Leftrightarrow$ Root of $\chi_T$
i.e if $\lambda$ is the root of minimal polynomial of T
iff $\lambda$ is the eigenvalue of T.

let P be the minimal polynomial of T.
suppose $P(\lambda) = 0$
then
$$P(t) = (t-\lambda)q(t)$$
then $P(T) = (T-\lambda I)q(T)$
$= 0$
$(T-\lambda I)\ q(T) = 0$
as $\deg(q) < \deg(P)$
$q(T) \neq 0$
as P is the mininum deg polynomical with
$P(T) = 0$

$\therefore$ $q(T)$ is a non-zero matrix

as $q(T)$ is a non-zero matrix
take a non-zero $0 \neq h \in \mathbb{C}^n$ s.t
$q(T)h = x$ is non-zero
so
$$(T-\lambda I)q(T)h = 0 \cdot h = 0$$
$\Rightarrow (T-\lambda I)x = 0$ for some $x \neq 0$
$\ker(T-\lambda I) \neq \{0\}$
$\therefore \det(T-\lambda I) = 0$
$\Rightarrow \lambda$ is a root of $\chi_T$

$\therefore \lambda$ is an eigenvalue

conversely let $\lambda$ be an eigenvalue of T
as $\sigma(P(T)) = P(\sigma(T))$
$\lambda \in \sigma(T)$

$\sigma(0) = 0$ as eigenvalues of 0 matrix s.t
$0x = \lambda x$
$0 = \lambda x \Rightarrow \lambda = 0$
for $x \neq 0$

Note : $P(T) = 0$ (0 matrix)
$\longrightarrow$ $\sigma(0) = 0$ (0 matrix Spectrum)
$\{0\} = \sigma(P(T)) = P(\sigma(T))$
this means that
$P(\lambda) = 0$
as $\lambda \in \sigma(T)$

$\therefore \lambda$ is a root of P.

Note : If $T \in M_{n\times n}(\mathbb{C})$ has n distinct eigenvalues $\lambda_1 \dots \lambda_n$ then
$P(t) = (t-\lambda_1)\cdots(t-\lambda_n)q(t)$ $\leftarrow$ will have a root but
$P(t) = (t-\lambda_1)\cdots\cdots q(t)$ $\therefore$ const $\leftarrow$ not a polynomial

$$p(t) = q_0 \chi_T(t)$$
but
$q_0 = 1$ for $p$ to be monic

$\therefore\ p(t) = (t-\lambda_1)(t-\lambda_2)\cdots\cdots(t-\lambda_n)$

$\uparrow$
monic polynomial (minimal for $T$)

**Note**: If all $\lambda_i$ are distinct, then $T$ is a diagonisable matrix

$\therefore$ If $T$ is <u>diagonisable</u> then $P(t) = (t-\lambda_1)(t-\lambda_2)\cdots\cdots(t-\lambda_n)$

**prop**: minimal polynomial of $T$ = minimal polynomial of $R$ if $T = SRS^{-1}$ where $S \in M_{n\times n}(\mathbb{C})$

$T$ is similar to $R$ $(T \sim R)$

**proof**: suppose $P^{-1}TP = T'$ (i.e $T \sim T'$)

if $f(T') = 0$ → Anihates $T'$

then $f(P^{-1}TP) = f(T') = 0$
$= P^{-1} f(T) P$

$\left( \begin{array}{l} (P^{-1}TP)^n = P^{-1}T^nP \quad \forall n \in \mathbb{N} \\ \text{eg. } P^{-1}TP\cancel{P^{-1}}TP = P^{-1}T^2P \end{array} \right)$

as $P$ is invertible, so is $P^{-1}$.

now $f(T') = 0 \quad \forall f \in \mathbb{C}[Z]$
$\Rightarrow f(T) = 0$

$\therefore$ if $P_{T'}(T') = 0$
then $P_{T'}(T) = 0$
or $\langle P_{T'} \rangle \subseteq \langle P_T \rangle$

similary $\langle P_T \rangle \subseteq \langle P_{T'} \rangle$

or $\langle P_T \rangle = \langle P_{T'} \rangle$

as $P$'s are unique

$\Rightarrow P_T = P_{T'}$ say $P$

or minimal polynomial of $T$ = minimal polynomial of $T'$

**Note**: If $T$ is diagonisable, then $p(t) = (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_m)$ ← minimal polynomial of $T$
where $\lambda_1, \lambda_2, \ldots \lambda_m$ are distinct eigenvalues

As similar matrices have same minimal polynomial.

$$T \sim \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

as $T$ is diagonsible, it is similar to $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

minimal polynomial of $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ $= (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)$ → distinct ones that appear
$= p(t)$
= minimal polynomial of T

$\therefore$ minimal polynomial of $T = (t-\lambda_1)(t-\lambda_2)\cdots\cdots(t-\lambda_n)$

Note : No need to take power, if we only take diff eigenvalues we find minimal polynomial.

Theorem : (Caley - Hamilton theorm)
$\chi_T(T) = 0$ for any $T \in M_{n\times n}(\mathbb{C})$

( $\chi_T$ anilates $T$ or $\chi_T \in \langle p \rangle$ where $p$ = minimal polynomial of $T$ )

Note : $p(t) = \sum_{n=0}^{m} A_n t^n$ → coeffs are matrices of fixed size
$p(t)$ → matrix valued polynomials

$p(T) = \sum_{n=0}^{m} A_n T^n$ where $T^0 = I$
→ same order as coefficients

$q(t) = \sum_{n=0}^{k} B_n t^n$ another such matrix valued polynomial

$pq(t) = \sum_{n=0}^{m+k} C_n t^n$ where $C_n = \sum_{r+s=n} A_r B_s$

$pq(T) = \sum_{n=0}^{m+k} C_n T^n$
$= p(T)\, q(T)$

if $T$ commutes with $B_n$'s

$p(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$

becaus all polynomials, collect all coef of say $x^t$
→ gives us matrix

so $p(t) = \sum_{n=0}^{m} A_n t^n$

proof : Take $p(s) = (T - sI)$
→ matrix valued polynomial

let $q(S)$ be the Adj $P(t)$

is also a matrix valued polynomial

$q(S)\,P(S) = \chi_T(S)\, I$ ← Roots of $\chi_T$ are roots of minimal polynomial

coeff of $P(S)$ is either in $T$ or in $T-S^-$

$\Rightarrow q(T)\,P(T) = \chi_T(T).\, I$ as minimal poly $\times h(S) = q(S) \times p(S)$

maximal polynomial

$\Rightarrow \chi_T(T) = 0$

$q(T)\,P(T) = 0$

or $\chi_T(T) = 0$

Note: if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then $C_{31} = (-1)^{3+1} A_{31}$

$c_{ij} = (-1)^{i+j} A_{ij}$

$\mathrm{Adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

and $A \,\mathrm{adj}\, A = |A|\, I$

or $\frac{1}{|A|} \mathrm{Adj}(A) = A^{-1}$

3rd oct :

finite dimensional vector space :

let $V$ be a finite dimensional vector space, and let $W_i$ be subspaces of $V$. Then $W = \underbrace{W_1 + \dots W_n}_{\text{subspaces}}$

this is an subspace of $V$. (satisfies $u \in W, v \in W$ then $u + \alpha v \in W$)

(Direct sum : every vector in $W$, can be expressed uniquely as sum of $w_1, w_2, \dots w_n$)
if

Defn: (Equivalent to direct sum)
we say $\{W_i\}$ are independent if $x_i \in W_i$ and

$$x_1 + x_2 + \dots x_n = 0 \Rightarrow x_i = 0 \quad \forall i$$

Prop: $\{W_1, \dots W_n\}$ is independent iff $W = W_1 \oplus W_2 \oplus \dots \oplus W_n$

proof :

$(\Rightarrow)$ $x_1 + x_2 + \dots x_n = y_1 + y_2 + \dots y_n$

then

$$(\underbrace{x_1 - y_1}_{\in W_1}) + \dots (\underbrace{x_n - y_n}_{\in W_n}) = 0$$

$\Rightarrow x_1 = y_1, \dots x_n = y_n$

or $x_i = y_i \quad \forall i$

$\therefore$ unique combination

$\therefore W = W_1 \oplus W_2 \oplus \dots W_n$

$(\Leftarrow)$ if direct sum, every vector uniquely rep, then it is trivial that

if $x_i \in W_i$

then $x_1 + x_2 + \dots x_n = 0$

$\Rightarrow x_1 = 0$

$x_2 = 0$

$\vdots$

$x_n = 0$

or $\{W_i\}$ is independent

Lemma: let $V$ be a f.d vector space and $W_i$'s are subspaces of $V$, and $W = W_1 + \dots W_n$

TFAE :

(i) $W = W_1 \oplus W_2 \oplus \dots \oplus W_n$

(ii) for all $2 \le j \le n$,
$(W_1 + \dots + W_{j-1}) \cap W_j = \{0\}$

(iii) If $B_1 \dots B_n$ are basis of $W_1, \dots W_n$ respectively then $\bigcup_{i=1}^{n} B_i$ is a basis for $W$.

(iv) $\dim(W) = \sum_{i=1}^{n} \dim(W_i)$

proof: (i) $\Rightarrow$ (ii):
suppose that there is a non-zero vector
$0 \neq x \in W_j \cap (W_1 + W_2 + \cdots W_{j-1})$

$\Rightarrow \underbrace{0+0+\cdots+x}_{j^{th}\text{ place}}+0 = y_1+y_2+\cdots y_{j-1} + 0+0+\cdots 0$ +0. 0

$\Rightarrow 0+0+\cdots 0+x+0+0\cdots 0 = y_1+y_2+\cdots y_{j-1}+0+0+\cdots 0$
$\Rightarrow \quad x = 0$
as $W = W_1 \oplus W_2 \oplus \cdots \oplus W_n$
$\{W_i\}$ is unique
or $y_1+y_2+\cdots y_{j-1} - x_j = 0$
$\Rightarrow$
$y_1 = y_2 = \cdots = y_{j-1} = 0$
$\therefore \quad x \in \{0\}$
or $\{0\} = W_j \cap (W_1 + \cdots W_{j-1})$

(ii) $\Rightarrow$ (i)

suppose $\alpha_1+\cdots+\alpha_n = \beta_1+\cdots\beta_n$ where
$\alpha_i, \beta_i \in W_i, \forall i = 1,2,\ldots,n$

$\alpha_n - \beta_n = (\beta_1-\alpha_1)+(\beta_2-\alpha_2)+\cdots$
$\cdots(\beta_{n-1}-\alpha_{n-1})$

now as $W_n \cap (W_1+W_2+\cdots W_{n-1}) = \{0\}$
we have $\alpha_n = \beta_n$
and $\alpha_1+\alpha_2+\cdots\alpha_{n-1} = \beta_1+\beta_2+\cdots\beta_{n-1}$
iterating this to get
$\alpha_{n-1} = \beta_{n-1}$
$\vdots$
$\alpha_1 = \beta_1$

or $\alpha_i = \beta_i \quad \forall i$
$\therefore \{W_i\}$ is independent
$\Rightarrow W = W_1 \oplus W_2 \oplus \cdots \oplus W_n$

(iii) $\Rightarrow$ (i):

As $B_1, B_2, \ldots B_n$ are basis and
as $\cup B_i$ is a base of $W$
or $\cup B_i$ is independent
and $\cup B_i$ spans $W$

now, as $\cup B_i$ is independent
all $x_i \in W_i$ can be
rep as $x_i \in \operatorname{span} B_i$
then all $x_i$ are lin ind
$\therefore \{W_i\}$ is ind $\Rightarrow W = W_1 \oplus W_2 \oplus \cdots \oplus W_n$

(i) $\Rightarrow$ (iii) : $B = \bigcup_{i=1}^{n} B_i$ spans $W$ (trivial)

so if $\bigcup_{i=1}^{n} B_i$ is independent, then we are done.

To show: $\bigcup_{i=1}^{n} B_i$ is ~~linearly ind~~.

let $B_i = \{b_{1i}, b_{2i}, \ldots \ldots b_{r_i i}\}$ $\forall\ i = 1, 2, \ldots \ldots n$

now if $\sum_{i=1}^{n} \sum_{j=1}^{r_i} b_{ji}\, \alpha_{ji} = 0$

$$= \left(\sum_{j=1}^{r_1} \alpha_{j1} b_{j1}\right) + \left(\sum_{j=1}^{r_2} \alpha_{j2} b_{j2}\right) + \ldots \left(\sum_{j=1}^{r_n} \alpha_{jn} b_{jn}\right)$$

$0 = x_1 + x_2 + \ldots \ldots x_n$, $x_i \in W$

as $W = W_1 \oplus W_2 \oplus \ldots W_n$ (as $\{W_i\}$ is ind)
$\Rightarrow$ all $x_i = 0$

now as all $x_i = 0$

$\sum \alpha_{ji} b_{ji} = 0$

$\Rightarrow \alpha_{ji} = 0\ \forall\ i, j$

as $b_{ji}$ are lin ind wrt $i$

(iii) $\Leftrightarrow$ (iv) is trivial

**Defn**: A linear map $E$ on a vector space $V$ is a projection if $E^2 = E$

**Note**: $x \in \operatorname{Ran} E$ (Range of $E$) $\Leftrightarrow$ $E(x) = x$

**proof**: ($\Rightarrow$) $x \in \operatorname{Ran} E$
then $\exists\ x' \in \operatorname{dom}(E)$ s.t $E(x') = x$

also as $E^2 = E$
$E(E(x')) = E(x')$
$\Rightarrow E(x) = x$

($\Leftarrow$) If $E(x) = x$ then
as $x \in \operatorname{dom}(E)$ and $x = E(x)$
$\Rightarrow x \in \operatorname{Ran}(E)$

$\therefore\ x \in \operatorname{Ran}(E) \Leftrightarrow E(x) = x$

**Note**: As orthocompliment of $x \in \operatorname{Ran}(E)$ is NE
we have other then $E(x) = x$, every other $x$'s s.t $E(x) \neq x$

then $E(x) = 0$.

so either $E(x) = x$ $\quad x \in \text{Ran } E$
or
$E(x) = 0$ $\quad x \in N_E$

$\therefore$ $\text{Ran } E \oplus \text{Null } E = V$

Note : $\text{Ran} E \oplus \text{Null } E = V$ $\left( \begin{array}{l} \text{Ran } E \oplus V_1 = V \\ \dim V_1 = \dim(\text{Null } E) \end{array} \right.$ $\left. \begin{array}{l} \text{Null}(E) \cap \text{Ran} E = \{0\} \\ \text{Null}(E) \oplus \text{Ran} E = V \end{array} \right)$

let $W$ be subspace of $V$, then what is the projection whose range is $W$ :

idea : we define map $F$ from whose space $V$ and let $\{w_1, \dots w_m\}$ basis for $W$
$\{w_1, w_2, \dots w_m, r_1, r_2, \dots r_n\}$ basis for $V$

then $F$ s.t $F(w_i) = w_i \quad \forall i$
$F(r_j) = 0 \quad \forall j$

$v \in V \Rightarrow v = \sum \alpha_i w_i + \sum \beta_j r_j$
$F(v) = F(\sum \alpha_i w_i) + F(\sum \beta_j r_j)$
$F(v) = F(\sum \alpha_i w_i) = \sum \alpha_i w_i$

now $w \in W$
$F(w) = w$
and $v \in V \setminus W$
$F(v) = 0$

so we can find a projection ($F$, which is one-one)

Note : The map $E \longrightarrow \text{Ran } E$ is a bijective map is a bijective map between collections of all projections on $V$ and subspaces of $V$.

lemma : let $V$ be a finite dimentional vectorspace and $W_i$'s be subspaces of $V$. Suppose

$$V = W_1 + W_2 + \cdots W_n$$

TFAE :

(i) $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$

(ii) $\exists$ projection $E_1, \dots E_n$ with $\text{Ran } E_i = W_i$ s.t
 (a) $E_i E_j = 0, \forall i \neq j$
 (b) $I = E_1 + E_2 + \cdots E_n$

(Note : $E_i(I) = \underset{\to 0}{E_i E_1} + \underset{\to 0}{E_i E_2} + \cdots E_i E_i + \cdots \underset{\to 0}{\cdots}$
or $E_i = E_i^2$ )

proof : (i) $\Rightarrow$ (ii) :

$$V = W_1 \oplus W_2 \oplus \cdots W_n$$

as $V$ is direct sum

$7^{th}$ oct :

Recall : $E : V \to V$ s.t

$E^2 = E \leftarrow$ projection

(i) $E(x) = x \iff x \in \text{Ran}(E)$

(ii) $V = \text{Ran}\, E \oplus \text{Null}\, E$

Note: The previous lemma was not correct and also the fact $E \leftrightarrow \text{Ran}\, E$ is false

Note: $E \mapsto \text{Ran}\, E$ is surjective (not one-one)

lemma: let $V$ be a vector space and $W_1, W_2, \ldots W_n$ be subspaces of $V$. TFAE:

(i) $V = W_1 \oplus \cdots \oplus W_n$

(ii) $\exists$ linear maps $E_i$'s $i = 1, 2, \ldots n$ s.t

(a) $I = E_1 + E_2 + \cdots E_n$

(b) $E_i E_j = 0 \ \forall i \neq j$

(c) $E_i^2 = E_i \ \forall i$

(d) $\text{Ran}\, E_i = W_i, \ \forall i = 1, 2 \ldots n$

proof: $((2) \Rightarrow (1))$ since $I = E_1 + E_2 + \cdots E_n$

$v = E_1(v) + E_2(v) + \cdots E_n(v) \quad \forall v \in V$

$\Rightarrow v \in W_1 + W_2 + \cdots W_n$

for $v \in V$, suppose it has 2 diff expression

i.e $v = \Sigma \alpha_i = \Sigma \beta_i$

for $\alpha_i, \beta_i \in W_i \ \forall i = 1, 2, \ldots n$

$\Sigma \alpha_i \in W_1 + W_2 + \cdots W_n$
$= \Sigma \beta_i \in W_1 + W_2 + \cdots W_n$

$E_i(v) = E_i(\alpha_1) + \cdots E_i(\alpha_i) + \cdots E_i(\alpha_n)$
$= E_i(\beta_1) + \cdots E_i(\beta_i) + \cdots E_i(\beta_n)$

or $E_i(\alpha_i) = E_i(\beta_i)$

and $E_i(v) = E_i(\alpha_i) = E_i(\beta_i)$
$= \alpha_i = \beta_i$

$(\because E_i(\alpha_i) = \alpha_i \iff \alpha_i \in \text{Ran}\, E_i)$

$\therefore \alpha_i = \beta_i \ \forall i$

now as $\text{Ran}\, E_i = W_i$

$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$

$((1) \Rightarrow (2))$

Here if $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$

( $E \mapsto \text{Ran}\, E$ is surjective as $W \subseteq V$, basis $W = \{\alpha_1, \alpha_2, \ldots \alpha_d\}$
$V = W \oplus W'$, basis $W' = \{\beta_1, \beta_2, \ldots \beta_r\}$

$E$ is s.t $E(\alpha_i) = \alpha_i$
$E(\beta_j) = 0$ then $E^2 = E$ )

(Here $\exists E$ s.t $E^2 = E$ for given Rom E)

now $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$

then $W_1' = W_2 \oplus \cdots \oplus W_n$

for $1 \leq i \leq n$,
since $V = W_i \oplus (W_1 \oplus \cdots \oplus W_{i-1} \oplus W_{i+1} \cdots \oplus W_n)$

define $E_i : V \to V$ by
$E_i(f) = f$, $\forall f \in W_i$
$E_i(g) = 0$, $\forall g \in (W_1 \oplus \cdots W_{i-1} \oplus W_{i+1} \cdots \oplus W_n)$

we can extend $E_i$ linearly to get:
$E_i(f+g) = E_i(f) + E_i(g)$
or $E_i$ is a linear map

Now Range of $E_i = W_i$ (d)
and
$E_i^2 = E_i$ (c)

also, $E_i E_j = 0$ as
rom $E_j = W_j$
and $W_j \subseteq$ ker $E_i$ (b)

now to show $I = E_1 + E_2 + \cdots E_n$

$v \in V$

$(E_1 + E_2 + \cdots E_n)(v) = E_1(v) + \cdots E_n(v)$

as $V = W_1 \oplus \cdots W_n$

$v = w_1 + w_2 + \cdots w_n$ uniquely for $w_i \in W_i$

$$E_i(v) = E_i(w_1 + \cdots w_n) = E_i(w_i) = w_i$$

so $(E_1 + \cdots E_n)(v) = w_1 + \cdots w_n = v$
$\forall v \in V$

$\therefore E_1 + \cdots E_n = I$

Note: $E^2 = E$ and $V = \text{Ran } E \oplus \text{Null } E$

Basis for Ran(E) $\{\alpha_1 \ldots \alpha_d\}$ $\cup$ Basis of Null(E) $\{\beta_1 \ldots \beta_r\}$ → Basis of V

$E : V \to V$
$E(\alpha_i) = 1\alpha_1 + (0)\alpha_2 + (0)\alpha_3 + \cdots 0$
$E(\beta_i) = 0 = 0.\alpha_1 + 0\alpha_2 + \cdots 0$

or $E = \left[\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right]_{d+r \times d+r} = \begin{bmatrix} I_{d\times d} & 0 \\ 0 & 0 \end{bmatrix}_{d+r \times d+r}$

$\underbrace{\phantom{xxxx}}_{d}$ $\underbrace{\phantom{xx}}_{r}$

$e_1, e_2, \ldots e_d$ $\quad r \to$ all zero

**Note**: $V = W_1 \oplus W_2$, $T : V \to V$

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\left.\begin{array}{l} A \in \alpha(W_1, W_1) \\ B \in \alpha(W_2, W_1) \\ C \in \alpha(W_1, W_2) \\ D \in \alpha(W_2, W_2) \end{array}\right\} \text{linear maps}$$

**Defn**: let $T : V \to V$ be a linear map and A subspace $W$ of $V$ is called invariant subspace for $T$ if

$$T(W) \subseteq W$$

**Note**: if $W$ is invariant for $T$ then $T = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ $\quad V = W \oplus W'$

$C \in \alpha(W_1, W_2)$
as $W_1 = W$
$W_2 = W'$

as $T(W) \subseteq W$
$C$ has to be zero

as if basis $W = \{\alpha_1, \ldots \alpha_d\}$ $\quad W' = \{\beta_1 \ldots \beta_s\}$

$T(\alpha_i) \in W$ so

$$T(\alpha_i) = *\alpha_1 + *\alpha_2 + \cdots *\alpha_d + \underbrace{(0)\beta_1 + 0 + 0 \cdots}_{C=0}$$

$$T = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

If $W'$ is also invariant then $\underset{=0}{B \in \alpha(W_2, W_1)}$

or $T = \left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right]$

as $T(\beta_i) = 0\alpha_1 + \cdots 0\alpha_d + *\beta_1 + \cdots *\beta_s$

**Theorm:** let $T$ be a linear map on $V$, and let $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$ and let $E_i$'s be projections s.t

(a) $I = E_1 + \cdots E_n$
(b) $E_i E_j = 0$, $\forall i \neq j$
(c) $W_i = \text{Ran } E_i$, $\forall i = 1, 2, \ldots n$

**TFAE:**

(i) Each $W_i$ is an invariant subspace of $T$

(ii) $TE_i = E_i T$, $\forall i = 1, 2, \ldots n$

**proof:** $(ii) \Rightarrow (i)$ $TE_i = E_i T$ $\forall i = 1, 2, \ldots n$

for $\alpha_i \in W_i$

$$T(\alpha_i) = T(E_i(\alpha_i)) = E_i(T(\alpha_i))$$

$$T(\alpha_i) = E_i(T(\alpha_i))$$

$$\Leftrightarrow T(\alpha_i) \in \text{Ran } E_i$$
$$\Rightarrow T(\alpha_i) \in W_i$$

so $\forall \alpha_i \in W_i$
$T(\alpha_i) \in W_i$
or $W_i$ is invariant

$((i) \Rightarrow (ii))$ If All $W_i$ are invariant then for $v \in V$
s.t $v = w_1 + \cdots w_n$

$$TE_i(v) = T(w_i) \in W_i$$
as $W_i$ is invariant

$$= E_i(T(w_i)) \text{ as } T(w_i) \in W_i = \text{Ran } E_i$$

Now as $E_i(T(w_1 + w_2 + \cdots w_n))$

$$= E_i(T(w_1) + \cdots T(w_n))$$
$$= E_i(T(w_i))$$

so $TE_i(v) = E_i(T(w_i)) = E_i(T(v))$

**Note:** $T = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{bmatrix}$, $A_j \in \alpha(W_j, W_j)$

$A_j = T|_{W_j}$

10th oct :

**Theorm**: let $A \in \mathcal{L}(V,V)$ finite d vetor space, and let $\alpha_1, \dots \alpha_d$ be distinct eigenvectors of $A$. let $W_i$ be the eigenspace corresponding to $\alpha_i$ $i = 1,2,\dots n$

$(\ker(A-\alpha_i I) = W_i$

TFAE

(i) $A$ is diagonisable

(ii) the char. polynomial of $A$ has form

$$\chi_A(t) = (t-\alpha_1)^{n_1} \cdots (t-\alpha_d)^{n_d}$$

and $\dim(W_i) = n_i \quad \forall i = 1,2,\dots d$

(iii) $\dim(V) = \dim(W_1) + \dots + \dim(W_d)$

$\iff V = W_1 \oplus W_2 \oplus \dots W_d$

(iv) $\exists E_i \in \mathcal{L}(V,V)$, $(i = 1,2,\dots d)$ s.t

(a) $I = E_1 + \dots + E_d$
(b) $A = \alpha_1 E_1 + \dots \alpha_d E_d$
(c) $E_i E_j = 0 \quad \forall i \neq j$
(d) $E_j^2 = E_j \quad \forall j$
(e) Row $E_j = W_i$, $\forall i = 1,2,\dots d$

(v) The minimal polynomial of $A$ is

$$(t-\alpha_1) \cdots (t-\alpha_d)$$

**proof**: $((i) \Rightarrow (ii))$ as $A$ is diagonisable

$S^{-1} A S = D$

and $\chi_A = \chi_D$

$$= (t-\alpha_1)^{n_1} (t-\alpha_2)^{n_2} \cdots (t-\alpha_d)^{n_d}$$

repeated in term of multiplicity

$$\underbrace{\begin{bmatrix} \alpha_1 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix}}$$

$n_1$ times

eigenvtos of this metrics are $e_i$

$\Rightarrow e_1$ to $e_{n_1}$ are eigenvetos corresponding to $\alpha_1$

$\Rightarrow \dim(W_1) = n_1$

generally $\dim(W_i) = n_i$
wrt $D$

$$\chi_D(t) = \prod (t-\alpha_i)^{n_i}$$

$\dim(W_i) = \dim(\text{eigenspace wrt } \alpha_i \text{ in } D)$ $(\because A \sim D)$
wrt $A$

$\dim(W_i) = n_i$
$\hookrightarrow$ wrt $A$

$\big((ii) \Rightarrow (iii)\big)$ By 2, $n_1 + n_2 + \cdots n_d = \dim(V)$
$\dim W_i = n_i$

or $\sum \dim W_i = \dim(V)$

Now as $W_i$ and $W_j$ are ind for $i \neq j$
(eigenvectors corr to diff eigenvalues)

so $V = W_1 \oplus W_2 \oplus \cdots W_d$

$\big((iii) \Rightarrow (iv)\big)$ as $V = W_1 \oplus W_2 \oplus \cdots W_d$
$\exists$ projections $E_i$'s s.t
(a), (c), (d), (e) hold

let $v \in V$ then as $V = W_1 \oplus \cdots \oplus W_d$

$v = \sum_{i=1}^{d} v_i$ for $\forall v_i \in W_i$
and is unique

now, $A(v) = A\left(\sum_{i=1}^{d} v_i\right) = \sum_{i=1}^{d} A(v_i)$

$= \sum_{i=1}^{d} \alpha_i (v_i)$

$A(v) = \sum_{i=1}^{d} \alpha_i E_i (v_i)$

$= \sum_{i=1}^{d} \alpha_i E_i \left(\sum v_i\right)$ as $E_i$ kills all other vectors

$A(v) = \sum_{i=1}^{d} \alpha_i E_i (v)$

$\Rightarrow A = \sum_{i=1}^{d} \alpha_i E_i$

(Note: $V = W_1 \oplus W_2 \oplus \cdots W_d$
then $A = \begin{bmatrix} \alpha_1 I_{W_1} & & \\ & \alpha_2 I_{W_2} & \\ & & \ddots \end{bmatrix}$

$E_1 = \begin{bmatrix} I & & 0 \\ & 0 & \\ 0 & & \ddots 0 \end{bmatrix}$ $E_2 = \begin{bmatrix} 0 & & 0 \\ & I & \\ 0 & & 0 \ddots 0 \end{bmatrix}$ )

Remarks : (a), (b), (c) $\Rightarrow$ Range $E_j = W_j$ (e)

If we take $v_j \in$ Ran $E_j$, then

$$A(v_j) = \sum \alpha_i E_i(v_j) = \alpha_j E_j(v_j)$$

$$A(v_j) = \alpha_j v_j \Rightarrow v_j \in W_j$$

$$\text{Ran } E_i \subseteq W_i \quad \forall j$$

and if $v_j \in W_j$ i.e $A(v_j) = \alpha_j v_j$

$$\Rightarrow \sum_{i=1}^{d} \alpha_i E_i(v_j) = \alpha_j v_j = \alpha_j \left( \sum_{i=1}^{d} E_i v_j \right)$$

from (b)     from (a)

$$\Rightarrow \sum \alpha_i E_i(v_j) = \alpha_j \left( \sum E_i v_j \right)$$

$$\Rightarrow \sum (\alpha_i - \alpha_j) E_i(v_j) = 0$$

$\Rightarrow$ if we apply $E_1$ on both sides we get

$$(\alpha_1 - \alpha_j) E_1^2 (v_j) = 0 \Rightarrow E_1(v_j) = 0$$

or $\forall i \neq j$ $\quad E_i(v_j) = 0$

also $\quad v_j = \sum E_i v_j = E_j v_j$

$$\Rightarrow v_j = E_j v_j$$

$$\Rightarrow v_j \in \text{Ran } E_j$$

$$\Rightarrow W_j \subseteq \text{Ran } E_j$$

$\therefore \quad W_j = \text{Ran } E_j$

((iv) $\Rightarrow$ (i)) As (a), (c), (e) is there it implies

$$V = W_1 \oplus \cdots \oplus W_d$$

Therefore A has suff eigenvectors which span V

Thus A is diagonalisable

((iv) $\Rightarrow$ (v)) by (b) $A = \alpha_1 E_1 + \alpha_2 E_2 + \cdots \alpha_d E_d$ where $E_i$ are projections

$$\Rightarrow A^2 = \sum_{i=1}^{d} \alpha_i^2 E_i$$

any polynomial $g \in \mathbb{C}[x]$

$$g(A) = \sum_{i=1}^{d} g(\alpha_i) E_i$$

$\left( \underline{\text{Note}} : g(t) = \sum \alpha_i t^i \Rightarrow g(A) = \sum \alpha_i A^i \right)$

<u>claim</u>: $g(A) = 0 \Leftrightarrow g(\alpha_i) = 0 \quad \forall i = 1, \dots d$

($\Leftarrow$) Trivial

($\Rightarrow$) $g(A) = 0$ then $\sum_{i=1}^{d} g(\alpha_i) E_i = 0$

multiply by $E_i$

$\Rightarrow g(\alpha_i) = 0 \quad \forall i = 1, \dots d$ ($\because E_i$ are not trivial)

so for $g(A) = 0$ $\Rightarrow$ $g(\alpha_i) = 0$
or $\alpha_i$ are the roots

$\therefore$ minimal polynomial with $\alpha_i$ roots for $i = 1, \dots d$ are:

$$\prod_{i=1}^{d} (t - \alpha_i)$$

14th oct :

**Theorem:** let $T \in \mathcal{L}(V)$

(v) The minimal polynomial of $T$ is $\prod_{i=1}^{d}(t-\alpha_i)$

(iv) $\exists\, E_i \in \mathcal{L}(V,V)$ s.t
(a) $I = E_1 + E_2 + \cdots E_n$
(b) $A = \alpha_1 E_1 + \cdots\cdots + \alpha_d E_d$
(c) $E_i E_j = 0 \;, \forall\, i \neq j$

**proof:** $((v) \Rightarrow (iv))$

$$p_1(t) = \prod_{i=2}^{d} \frac{(t-\alpha_i)}{(\alpha_1-\alpha_i)}$$

$$p_j(t) = \prod_{\substack{i=1,\\ i\neq j}}^{d} \frac{(t-\alpha_i)}{(\alpha_j-\alpha_i)}$$

(Lagrange polynomials)
(degree $d-1$)

**Note:** $p_j(t) = \begin{cases} 0 & ; \text{ else} \\ 1 & ; \ t = \alpha_j \end{cases}$

**Claim:** $p_1, \ldots p_d$ are linearly independent → do

then $1 = c_1 p_1 + c_2 p_2 + \cdots c_d p_d$
$t = b_1 p_1 + b_2 p_2 + \cdots b_d p_d$
$1 = c_1 p_1 + \cdots = p_1 + \cdots$
$t = b_1 p_1 + \cdots = \alpha_1 p_1 + \cdots$

so we are assuming $d > 1$, so there is nothing to prove for the case when $d=1$

($d=1$, it is a trivial case as only 1 eigenvalue, so that is the minimal poly $a = \alpha I$ → which is diagonal)

**Note:** $I = p_1(A) + p_2(A) + \cdots p_d(A)$

$A = \alpha_1 p_1(A) + \cdots\cdots \alpha_d p_d(A)$

set $E_i = p_i(A) \;, \forall\, i = 1 \ldots d$

then $E_i E_j = p_i(A) p_j(A) \quad i \neq j$
$= (p_i p_j)(A)$

$$p = (t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_d)$$

as $p_i p_j = \prod \frac{(t-\alpha_i)}{\alpha_i-\alpha_i} \prod \frac{(t-\alpha_i)}{\alpha_j-\alpha_i}$

we have $p \mid p_i p_j$

or $\Rightarrow$ as $p(A) = 0$
$\Rightarrow p_i p_j(A) = 0$

$\therefore\ i \neq j \Rightarrow E_i E_j = 0$ also (a), (b), (c) $\Rightarrow$ (e) (done) proof follows as $E_j = P_j(A) \neq 0$

generalised eigenvectors :

$V = W_1 \oplus \cdots\cdots \oplus W$

Theorm : Let $A \in \alpha(V,V)$ and let the minimal polynomial of $A$ be of the form

$p_1^{r_1} \cdots p_d^{r_d}$ (different) where $p_i$'s are irreducible monic polynomials over $\mathbb{F}$

then

(I) $V = W_1 \oplus \cdots\cdots \oplus W_d$, where
$W_i = \text{Null}(P_i^{r_i}(A))$, $i = 1, \ldots d$ ← elements in this are called generalised eigenvectors

(II) $W_i$ are invariant subspaces of $A$ ($T(W_i) \subseteq W_i$)

(III) The minimal polynomial of $T_i = A|_{W_i} : W_i \longrightarrow W_i$ is $p_i^{r_i}$

$$\left( A = \begin{bmatrix} T_1 & & & 0 \\ & T_2 & & \\ & & \ddots & \\ 0 & & & T_d \end{bmatrix} \text{ where } T_i \in \alpha(W_i, W_i) \right)$$
→ Block diagonal rep.

proof : set $f_i = \frac{P}{p_i^{r_i}}$, $\forall\, i = 1 \ldots d$

we want to write 1 like

$1 = f_1 g_1 + \cdots + f_d g_d$

this

look at ideal generated by $f_1, f_2 \ldots \ldots f_d$

ideals are principle so

$I_1 = (f_1)$

$\vdots$

$I_i = (f_i)$

$\vdots$

$I_d = (f_d)$

ideal generated by $(f_1, f_2 \ldots f_d) = I = \langle d \rangle$ ← as Ring is P I D

$= \{ f_1 g_1 + f_2 g_2 + \cdots f_d g_d \mid \forall\, g_i \in \text{Polynomials} \}$

and $d \mid f_i$, $\forall i = 1, 2 \ldots d$

some element in P.I.D

as $\langle d \rangle = I$

and $f_1 \in \langle d \rangle \Rightarrow f_1 = dk$ or $d \mid f_1$

fact : $d \mid f_i, \forall i = 1 \dots d \Rightarrow d \equiv 1$

so, $1 = f_1 g_1 + f_2 g_2 + \dots f_d g_d$

$\Rightarrow I = (f_1 g_1)(A) + \dots + (f_d g_d)(A)$

Define $E_j = f_j(A) g_j(A)$ , $\forall i = 1, 2 \dots d$

$E_i E_j = 0$ to prove they are projections

$E_i E_j = f_j(A) g_j(A) \; f_i(A) g_i(A)$

$= \underbrace{(f_i f_j(A))}_{\text{has a factor as minimal poly}}((g_i g_j)(A))$

$= 0$ as $p \mid f_i f_j$ , $i \neq j$

claim : Ran $E_i = W_i = \text{Null}(P_i^{r_i}(A))$

Ran $E_i \subseteq W_i$ as $P_i^{r_i}(A)[f_i(A) g_i(A) v]$

$= P(A)[g_i(A) v]$

$= 0$

if $x \in \text{Null}(P_i^{r_i}(A))$

$E_i(x_i) = f_i(A) g_i(A)(x_i)$

for $j \neq i$ $E_j(x_i) = f_j(A) g_j(A)(x_i)$

$= \underbrace{f_j(A)(x_i)}\, g_j(A)$

$= 0$ as $x_i \in \ker\{f_j(A)\}$

so $E_j(x_i) = 0$

$I = E_1 + \dots E_d$

$x_i$ on both sides we get

$x_i = E_i(x_i)$

$\Rightarrow x_i \in \text{Ran } E_i$

so Ran $E_i = W_i$

17th Oct :

**Theorem** : (Spectral theorem) Let $T$ be a linear map on a finite dim vector space $V$. Let

$$P = p_1^{r_1} \cdots p_d^{r_d}$$ be minimal polynomial of $T$ where $p_i$'s are distinct irreducible monic polynomial over $\mathbb{F}$ and $r_i$'s are positive integers.

suppose $W_i = \text{Null}(p_i^{r_i}(T))$, $i = 1, 2, \ldots d$

then

(i) $V = W_1 \oplus \cdots \oplus W_d$
(ii) $W_i$ is an invariant subspace of $T$, $\forall i = 1, 2 \ldots d$
(iii) min polynomial for

$$T_i = T\big|_{W_i} : W_i \to W_i \text{ is } p_i^{r_i}, \forall i = 1, 2 \ldots d$$

**proof** : set $f_i = \dfrac{P}{p_i^{r_i}}$, $\forall i = 1, \ldots d$

$$1 = \sum_{i=1}^{d} f_i g_i$$

or $1 \in \langle f_1, f_2 \ldots f_d \rangle$ (ideals generated by $f_1, \ldots f_d$)

any ideal here is principle (PID)

$$1 \in \langle f_1 \ldots f_d \rangle = \langle d \rangle$$

$$d \mid f_i \quad \forall i = 1, \ldots d$$

so $\quad d \mid \gcd(f_1, f_2, \ldots f_d)$ (as all $p_i$ are irred)

$$\Rightarrow d \mid 1 \Rightarrow d = 1$$

now as $\langle f_1, f_2, \ldots f_d \rangle = \langle 1 \rangle$

now, $\quad I = \sum_{i=1}^{d} f_i(T) g_i(T)$

set $\quad E_i = f_i(T) g_i(T)$

$E_i$ a projection as $E_i E_j = 0$

as : $E_i E_j = f_i(T) g_i(T) f_j(T) g_j(T)$
$= f_i(T) f_j(T) g_i(T) g_j(T)$
$= h(T) P(T) g_i(T) g_j(T)$
$= 0$

as $P(T) = 0$
$\uparrow$
minimal polynomial

**claim** : $\text{Ran } E_i = W_i$

if $x \in \text{Ran } E_i$ then

then

$x$ can be written as

$E_i(x) = x$ (as $E_i$ is a projection)

$\Rightarrow E_i(x) = f_i(T)\, g_i(T)\, x$

now, $p_i^{r_i}(T)\, x = p_i^{r_i}(T) E_i(x) = p_i^{r_i}(T)\, f_i(T)\, g_i(T)\, x$
$= p(T)\, g_i(T)\, x$
$= 0$

$\Rightarrow \operatorname{Ran} E_i \subseteq W_i$ $\left( \because p_i^{r_i}(T)\, x = 0 \Rightarrow x \in W_i \right)$

now, $E_i(W_j) = 0 \Rightarrow W_i \subseteq \operatorname{Ran} E_i$
for $i \neq j$

$x \in W_i$
$I = E_1 + \cdots E_d$
$x = E_1(x) + \cdots E_d(x)$
$\Rightarrow x = E_i(x)$
$\Rightarrow x \in \operatorname{Ran} E_i$
$\Rightarrow W_i \subseteq \operatorname{Ran} E_i$

to show $E_j(x) = 0$

as $x \in W_i = \operatorname{Null}\left(p_i^{r_i}(T)\right)$

$E_j(x) = g_j(T)\, f_j(T)(x)$
$= 0$

But $p_i^{r_i}(T)$ is in $f_j(T)$

So $V = W_1 \oplus \cdots \oplus W_d$ (i)
from previous theorems

for $x \in W_i = \operatorname{Null}\left(p_i^{r_i}(T)\right)$

$p_i^{r_i}(T)(Tx) = T\left(p_i^{r_i}(T)(x)\right) = 0$

$\Rightarrow T(W_i) \subseteq W_i \quad \forall\, i = 1, 2, \ldots d$
(ii)

for (iii) The minimal polynomial for $T_i = T\big|_{W_i}$

Restricting it to just act on $W_i$ and not whole of $V$

for $x \in W_i$ $\;p_i^{r_i}(T_i)\, x = p_i^{r_i}(T)\, x$
$= 0$ as $\forall\, j = 1, 2 \ldots d$
(shown that min x coming $= p_i^{r_i}(T)$) $\quad p_i^{r_i}(T_j) = 0$

so $p_i^{r_i}$ can be a minimal polynomial

Proof of $P_i^{r_i}$ is minimal:

let $g(T_i) = 0$

then $g(T) f_i(T) = 0$

$x \in W_i$ $\quad f_i(T) g(T) x = f_i(T) g(T_i) x$
$= 0$

$W_j = \text{Null}(P_j^{r_i}(T))$

for $x \in W_j$ $\quad g(T) f_i(T) x = 0 \quad j \neq i$

$\rightsquigarrow g(T) f_i(T) = 0$

$\Rightarrow P \mid g f_i \leftarrow$ as $g(T) f_i(T) = 0$

$P$ minimal polynomial

$\Rightarrow P_i^{r_i} f_i \mid g f_i$

$\Rightarrow P_i^{r_i} \mid g$

$\Rightarrow P_i^{r_i}$ is minimal polynomial

Note: when $\mathbb{F} = \mathbb{C}$

$P = (t - \lambda_1)^{r_1} \cdots (t - \lambda_d)^{r_d}$

Note: case when $T$ is diagonisable, all $r_i = 1$

$\therefore P = \prod (t - \lambda_i)$

here

$V = W_1 \oplus \cdots \oplus W_d$

space; eigenspaces & invariant

then $T = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{bmatrix}$

$A_i \in \mathcal{L}(W_i, W_i)$

Block representation of matrix:

$V = W_1 \oplus W_2$
s.t $W_1$ & $W_2$ are invariant

basis $W_1 = \{a_1, \dots a_r\}$
basis $W_2 = \{b_1, \dots b_m\}$

basis $V = \{a_1, \dots a_r, b_1, \dots b_m\}$

$T(a_1) = \alpha_{11} a_1 + \alpha_{21} a_2 + \cdots \alpha_{r1} a_r$
$+ 0 b_1 + \cdots 0 b_m$ as invariant

$T(a_r) = \alpha_{1r} a_1 + \cdots \alpha_{rr} a_r + \sum 0 \cdot b_i$

similarly $T(b_i) = 0 + \sum_{j=1}^{d} \beta_{ji} b_j$

then

$$T = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} & 0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rr} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \beta_{11} & \cdots \\ \vdots & \vdots & \vdots & \vdots & - & \\ 0 & 0 & 0 & 0 & - & \end{bmatrix} \rightarrow \text{not zero}$$

$$T = \begin{bmatrix} A_{r \times r} & 0 \\ 0 & B_{m \times m} \end{bmatrix}$$

$$T_A = T\big|_A = \begin{bmatrix} A_{r \times r} & 0 \\ 0 & 0 \end{bmatrix}$$

Note: the spectral theorm gives T like:

$$T = \begin{bmatrix} T_1 & & 0 \\ & T_2 \ddots & \\ 0 & & T_d \end{bmatrix}$$

$T_i = T\big|_{W_i} : W_i \rightarrow W_i$

in case of $\mathbb{F} = \mathbb{C}$ & and minimal polynomial

min of $T_i = (t - \lambda_i)^{r_i}$

note: $(T_i - \lambda_i I)^{r_i} = 0$

$N_i^{r_i} = 0$ (important operator)

when $N_i = T_i - \lambda_i I$
$\forall i = 1, 2 \ldots d$

so this theorm also tells us to understand this important operator

$N_i^{r_i} = 0 = (T_i - \lambda_i I)^{r_i}$

if $\mathbb{F} = \mathbb{C}$, $p = (t - \lambda_i)^{r_i} \ldots (t - \lambda_d)^{r_d}$

$W_i = \text{Null}\,(T - \lambda_i I)^{r_i}$

$N_m = \text{Null}\,(T - \lambda_i I)^m$ for $m \in \mathbb{N}$

Note: $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots \subseteq N_n = N_{n+1}$
as whatever vector A kills, is killed by $A^2$

property of P.I.D

$n =$ index of eigenvalue $\lambda_i$

Defn : index of eigenvalue $\lambda_i = n$

s.t

$$N_m = \text{Null}(T-\lambda_i I)^m \quad \& \quad N_n \neq N_{n-1}$$

$$N_1 \subseteq N_2 \cdots \subseteq N_n = N_{n+1} = \cdots$$

lemma : If $\lambda_i$ is a eigenvalue of $T$ & $n$ is the index of $\lambda_i$, then

then $p = \prod (t-\lambda_i)^{r_i}$ is the minimal poly

then $n = r_i$

proof : we have to show

$$\text{Null}(T-\lambda_i)^{r_i - 1} \neq \text{Null}(T-\lambda_i)^{r_i}$$

$$\& \ \text{Null}(T-\lambda_i)^{r_i} = \text{Null}(T-\lambda_i)^{r_i+1}$$

(By defination, this $r_i =$ index)

If $\text{Null}(T-\lambda_i I)^{r_i - 1} = W_i$

then

minimal polynomial for $T_i$ divides $(T-\lambda_i I)^{r_i - 1}$

which is not possible as minimal poly for $T_i$ is $(T-\lambda_i I)^{r_i}$

so $\text{Null}(T-\lambda_i I)^{r_i - 1} \neq W_i$

now $W_i = \text{Null}(T-\lambda_i I)^{r_i}$

if

$0 \neq y \in W_j$

then

$$\text{if} \quad (T-\lambda_i I)^{r_i+1} y = 0$$

then $(T-\lambda_i I)^{r_i} \underbrace{(T-\lambda_i I) y}_{v} = 0$

$\Rightarrow v \in W_i$

$$v = \underset{\in W_j}{Ty} - \underset{\in W_j}{\lambda_i y}$$

$\Rightarrow v \in W_i \cap W_j = \{0\}$ ($\because$ Direct sum)

$\Rightarrow v = 0$

$(T-\lambda_i I)y = 0 \Rightarrow y \in W_i$ $*$

$\therefore \text{Null}(T-\lambda_i I)^{r_i+1} = W_i$

21st oct :

lemma: (key lemma)
Let $T \in \mathcal{L}(V)$ be a linear map on a finite dim vector space. Suppose that $T^{m-1}(x) \neq 0$ and $T^m(x) = 0$ for some $x \in V$ & $m \geq 1$. Then

$(x, Tx, \dots T^{m-1}x)$ is lin ind

proof: If $(x, Tx, \dots T^{m-1}x)$ is dependent then

$$\sum_{i=0}^{m-1} \underbrace{\alpha_i T^i}_{} x = 0$$

for some $\alpha_i \in \mathbb{F}$ not all zero

now,
$$(\alpha_0 x + \alpha_1 Tx + \dots \alpha_{m-1} T^{m-1}x) = 0$$

$$\Rightarrow T^{m-1}(\alpha_0 x + \dots \alpha_{m-1} T^{m-1}x) = 0$$

$$\Rightarrow \alpha_0 T^{m-1}(x) = 0$$

$$\Rightarrow \alpha_0 = 0$$

if we repeate the process, we get $\alpha_i = 0 \ \forall i$

$\therefore$ lin independent

Defn: Let $T \in \mathcal{L}(V)$, where $V$ is a finite dimentional vector space we say $T$ is cyclic if $\exists x (\neq 0) \in V$ and a positive integer $m$ s.t

$$\text{Span}\{x, Tx, \dots\dots T^m x\} = V$$

In such a case, $x$ is called a cyclic vector for $T$.

Defn: A linear map $T \in \mathcal{L}(V)$ is nilpotent if $T^m = 0$ for some possible integer $m$. The index of nilpotency is the smallest integer $r$ s.t $T^r = 0$

denote it by $\eta(T)$

Prop: Let $T$ be a nilpotent map on $V$ and $\dim(V) = n$. Then $\dim(V) =$ index of nilpotent of $T \Leftrightarrow T$ is cyclic

proof: ($\Rightarrow$) as $\dim(V) =$ index of nilpoent of $T = n$ then

as $T^n = 0$, $T^{n-1} \neq 0$, $\exists x \in V$ s.t $T^{n-1}(x) \neq 0$ & $T^n(x) = 0$

then by key lemma

$\{x, Tx, T^2x, \dots T^{n-1}x\}$ = Basis of $V$ ($\because$ lin ind and $n$ in size)

span$\{x, Tx, \dots\dots T^{n-1}x\} = V$
& $\{x, Tx, \dots T^{n-1}x\}$ is lin ind

$\Rightarrow T$ is cyclic

($\Leftarrow$) Trivial

**Note**: cyclic-nilpotent maps are special

If $T$ is a cyclic nil map, then $\exists\, x \in V$ s.t $\{x, Tx, \dots, T^{n-1}x\}$ is a basis for $V$.

$$Tx = 0.x + 1(Tx) + 0(T^2x) + 0(T^3x) + \dots 0(T^{n-1}x)$$
$$T^2x = 0 + 0 + 1 + 0 \dots 0$$

$$T \sim \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & & \vdots & \vdots \\ 0 & 1 & & & \\ \vdots & 0 & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}}$$

last column

$$T(T^{n-1}x) = 0 + 0 \dots 0$$

Also called Jordan block

This is how $T$ looks like (similar)

so

$$T \sim \begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}$$

If $2\times2$ then $T \sim \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$$T^2 \sim \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow T^2 = 0$$

**Theorem**: (Nilpotent splitting theorem) let $T \in \alpha(V)$ be a nilpotent map and $r = \eta(T) < \dim(V)$

suppose $x \in V$ s.t $T^{r-1}(x) \neq 0$ and
$V_1 = \text{span}\{x, Tx, \dots, T^{r-1}x\}$

then $V_1$ is
1. An invariant subspace of $T$; $T(V_1) \subseteq V_1$
2. $V_1$ has a $T$-invariant complimentary subspace
$\exists\, W_1$ s.t
$$V = W_1 \oplus V_1$$
$$T(W_1) \subseteq W_1$$

we will prove this later

**Sme**: what happens if index of nilpotency $T < \dim(V)$ $\big(\eta(T) < \dim(V)\big)$

as $T \in \alpha(V)$, $\dim(V) = n > \eta(T) = r$

$$V = V_1 \oplus W_1$$

where $V_1 = \text{span}\{x, Tx, \dots T^{r-1}x\}$

$V_1 = \text{span}\{x_1, Tx_1, \dots T^{r-1}x_1\}$

where $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$ $\quad T_1 = T|_{V_1} : V_1 \to V_1$

$\eta(T_1) = \dim(V_1)$ $\quad T_2 = T|_{W_1} : W_1 \to W_1$

now $T_1 = T|_{V_1}$ ← This is a cyclic-nilpotent operator

now $T_2 = T|_{V_2}$ is also a nilpotent as it is a restriction of nilpotent map

note: $\eta(T_2) \leq \eta(T_1) = \eta(T)$

as $\eta(T_2) \leq \eta(T)$

as $T_2 = T|_{W_1}$ and as $\eta(T_1) = \eta(T)$

we have $\eta(T_2) \leq \eta(T_1)$

as for any $x \in W_1$, $T_2^{r}(x) = T^{r}(x) = 0$

$$\therefore \eta(T_2) \leq \eta(T)$$

Note: If $T_2$ is cyclic we are done, otherwise we can keep on using Nilpotent spliting theorm to get cyclic-nilpotents

i.e $\eta(T_2) = r_2 < \dim(W_1)$
then
$$W_1 = V_2 \oplus W_2$$
$$V_2 = \text{span}\{x_2, T_2 x_2, \dots, T_2^{r_2-1} x_2\}$$

and $W_2$ is a $T_2$-invariant subspace

$$V = V_1 \oplus V_2 \oplus W_2$$

cyclic-nilpotent

$$T = \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2' & 0 \\ 0 & 0 & T_3' \end{bmatrix}$$

$T_2' = T_2|_{V_2} : V_2 \to V_2$

$T_3' = T_2|_{W_2} : W_2 \to W_2$

cyclic nil potent

any nilpotent

we repeat this process. Since $V$ is finite dimensitional this process will end after finite number of steps.

Theorm: Every nilpotent map can be written as a direct sum of a cyclic nilpotent maps.

$$J_r = \begin{bmatrix} 0 & & 0 \\ 1 & 0 & \\ 0 & 1 & 0 \end{bmatrix}_{r \times r} \rightarrow \text{Jordan Block of } \eta(J_r) = r$$

proof: from the previous problem we get

$$T \cong \begin{bmatrix} J_{r_1} & & 0 \\ & J_{r_2} & \\ 0 & & \ddots J_{r_k} \end{bmatrix} \leftarrow J\text{-matrix}$$

$$r_1 \geq r_2 \geq \cdots \cdots \geq r_k$$

lemma: let $W$ be a subspace of $V$, and let $T \in \mathcal{L}(V)$. If $W$ is $T$-invariant then
$T^{-1}(W)$ is also $T$-invariant

$$T^{-1}(W) = \{x \in V \mid T(x) \in W\}$$

proof: To show
$$T(T^{-1}(W)) \subseteq T^{-1}(W)$$
well
$$T(T^{-1}(W)) \subseteq W$$
and as $W$ is invariant
$$T(W) \subseteq W$$
$$\Rightarrow W \subseteq T^{-1}(W)$$
$$\therefore T(T^{-1}(W)) \subseteq W \subseteq T^{-1}(W)$$

$$\Rightarrow T(T^{-1}(W)) \subseteq T^{-1}(W)$$
or $T^{-1}(W)$ is invariant

24th oct :

**Theorem** : (Nilpotent splitting theorem) Let $T \in \mathcal{L}(V)$ be a nilpotent operator with $\eta(T) < \dim(V)$, and let $V_1 = \text{span}\{x, Tx, \dots T^{m-1}x\}$ where $T^{(m-1)}x \neq 0$. Then $\exists$ a $T$-invariant subspace $W_1$ s.t $V = V_1 \oplus W_1$

**Lemma (I)**: If $W$ is a $T$-invariant subspace then $T^{-1}(W)$ is also $T$ invariant
$(T(W) \subseteq W \Rightarrow W \subseteq T^{-1}(W) \Rightarrow T(T^{-1}(W)) \subseteq W \subseteq T^{-1}(W))$ —— ①

**Lemma (II)**: $T^{-1}(T(W)) = W + \text{Null } T$ for every subspace $W$
$(W \subseteq T^{-1}(T(W))$ can be seen$)$

**proof** : $W + \text{Null}(T) \subseteq T^{-1}(T(W))$ is trivial to see $(\because \forall x \in \text{Null } T,\ T(\text{Null } T) = 0 \subseteq T(W) \Rightarrow \text{Null } T \subseteq T^{-1}T(W))$

now $\forall\ x \in T^{-1}(T(W))$
$\Rightarrow T(x) \in T(W)$
$\Rightarrow T(x) = T(v)$ for some $v \in W$
$\Rightarrow T(x - v) = 0$
$\Rightarrow x - v \in \text{Null } T$
$\Rightarrow x \in W + \text{Null } T$
$\therefore T^{-1}(T(W)) \subseteq W + \text{Null } T$
or
$T^{-1}(T(W)) = W + \text{Null } T$ —— ②

**proof** : The proof is by induction on $m$.

**step 1** : Suppose $T$ is an nilpotent operator with $\eta(T) = 1$. Then $V_1 = \text{span}\{x\}$. st $x \neq 0$
As this is a zero operator — 0 operator
any subspace is $T$-invariant.

$\therefore$ Any complementary subspace of $V_1$, then $W_1$ is invariant under $T$ and $V = V_1 \oplus W_1$

Suppose the theorem is true for any nilpotent operator if index is $m-1$

**step 2** : Let $T$ be a nilpotent operator with $\eta(T) = m$.
$V_1 = \text{span}\{x, Tx, \dots T^{m-1}x\}$
where $T^{m-1}(x) \neq 0$

Look at $T_1 = T\big|_{\text{Ran } T} : \text{Ran } T \to \text{Ran } T$
as $T_1^{m-1}(Ty) = T^{m-1}(Ty) = 0$ (Here $Tx = y \in \text{Ran } T$, $T_1 : \text{Ran } T \to \text{Ran } T$)

Note : $T_1^{m-2}(Tx) = T^{m-1}(x) \neq 0$

or $\eta(T_1) = m-1$

$T_1 = T\big|_{\text{Ran } T}$. Here index of nilpotency is $m-1$,

$\exists$ a $T$-invariant subspace $Y_1 \subseteq \text{Ran } T \subseteq V$ (using induction)
s.t $\text{Ran } T = Y_1 \oplus Y_2$

$Y_1 = \text{span}\{Tx, T^2x, \dots T^{m-1}x\}$
$= \text{span}\{y, Ty, \dots T^{m-2}y\}$

**claim** : $V = V_1 + T^{-1}(Y_2)$ (Here $T(V_1) = Y_1$, $T(V) = T(V_1) \oplus Y_2$)

$V = T^{-1}(\text{Ran } T) = T^{-1}(Y_1 \oplus Y_2) = T^{-1}(Y_1) + T^{-1}(Y_2)$
$= T^{-1}(T(V_1)) + T^{-1}(Y_2)$
③ —— $= V_1 + \text{Null } T + T^{-1}(Y_2) = V_1 + T^{-1}(Y_2)$
(from ②) (Note: $\text{Null } T \subseteq T^{-1}(Y_2)$)

claim 2: $V_1 \cap Y_2 = \{0\}$

$T(V_1 \cap Y_2) \subseteq Y_1 \cap Y_2 = \{0\}$

$\Rightarrow V_1 \cap Y_2 \subseteq V_1$ & $V_1 \cap Y_2 \subseteq \text{Null } T$

$\Rightarrow V_1 \cap Y_2 \subseteq V_1 \cap \text{Null } T$

$= \text{span}\{T^{m-1}x\} \subseteq Y_1$

& $V_1 \cap Y_2 \subseteq Y_2$

$\Rightarrow V_1 \cap Y_2 \subseteq Y_1 \cap Y_2$

$\Rightarrow V_1 \cap Y_2 = \{0\}$

$\big( T(V_1 \cap Y_2) = T(V_1) \cap T(Y_2)$
$= Y_1 \cap T(Y_2)$
$\subseteq Y_1 \cap Y_2 = \{0\}$
as $T(Y_2) \subseteq Y_2 \big)$

step 3: Now $V = V_1 + T^{-1}(V_2)$ ( $Y_2 \subseteq T^{-1}(Y_2)$, $V_1 \cap T^{-1}(Y_2) \subseteq T^{-1}(Y_2)$ so $\exists Z$ )

$Y_2 \oplus (V_1 \cap T^{-1}(Y_2)) \oplus Z = T^{-1}(Y_2)$

fact: $V = V_1 \oplus (Y_2 \oplus Z)$ (Note that this is true)

claim 3: $Y_2 \oplus Z$ is $T$-invariant

$T(Y_2 \oplus Z) \subseteq T(Y_2) + T(Z)$

we know $T(Y_2) \subseteq Y_2$ (already known)

& $T(Z) \subseteq T(T^{-1}(Y_2)) \subseteq Y_2$

$\Rightarrow T(Y_2 \oplus Z) \subseteq Y_2 \subseteq Y_2 \oplus Z$

$\Rightarrow T(Y_2 \oplus Z) \subseteq Y_2 \oplus Z$

$$T \cong \begin{bmatrix} J_{m_1} & & 0 \\ & \ddots & \\ 0 & & J_{m_n} \end{bmatrix}$$

cyclic

cyclic

$m_1 \geq m_2 \geq \cdots \geq m_k$

$\therefore V = V_1 \oplus (Y_2 \oplus Z)$

$= V_1 \oplus W_1$

s.t $W_1$ is $T$-invariant

now, $\dim(\text{Null } T) = k$

$A \in \mathcal{L}(V, V)$

$\chi_A(t) = (t-\lambda_1)^{d_1} \cdots (t-\lambda_r)^{d_r}$ (characteristic polynomial)

$p(t) = (t-\lambda_1)^{s_1} \cdots (t-\lambda_r)^{s_r}$ (minimal polynomial)

$$A \cong \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{bmatrix} \quad A_i = A|_{W_i}$$

$W_i = \text{Null}(A_i - \lambda_i)$

Since $(A_i - \lambda_i I)$ is a nilpotent operator of index $S_i$

& $$A_i - \lambda_i I = \begin{bmatrix} J_{i1} & & 0 \\ & \ddots & \\ 0 & & J_{ik} \end{bmatrix}$$

$$\Rightarrow A_i = \begin{bmatrix} J_{i1} + \lambda I & & 0 \\ & \ddots & \\ 0 & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 1\lambda_1 & & 0 \\ & \ddots & \\ 0 & & 1\lambda_i \end{bmatrix}$$ ← Jordan block corresp. to $\lambda_i$

28th oct :

Defn : Let $V$ be a vector space over $\mathbb{R}/\mathbb{C}$. Then an inner product on $V$ is a map

$$\langle \cdot,\cdot \rangle : V\times V \to \mathbb{R}/\mathbb{C}$$
$$(x,y) \to \langle x,y\rangle \quad \text{s.t}$$

(i) $\langle \alpha x+\beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$, $\forall \alpha,\beta\in \mathbb{R}/\mathbb{C}$, $x,y,z\in V$

(ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(iii) $\langle x,x\rangle > 0$, $\forall x\neq 0$

Remark : $\langle x, \alpha y+\beta z\rangle = \overline{\langle \alpha y+\beta z, x\rangle}$
$= \bar{\alpha}\overline{\langle y,x\rangle} + \bar{\beta}\overline{\langle z,x\rangle}$
$= \bar{\alpha}\langle x,y\rangle + \bar{\beta}\langle x,z\rangle$

Example : (i) $V=\mathbb{R}^n$, $\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i$ (dot product)

① trivial
② trivial
③ $\langle x,x\rangle > 0$ (as all squares)

$\therefore \langle x,y\rangle = \sum_{i=1}^{n} x_i y_i$ is an inner product

(ii) $V=\mathbb{C}^n$
$\langle x,y\rangle = \sum_{i=1}^{n} x_i \bar{y_i}$

① trivial
② $\sum x_i\bar{y_i} = \sum \overline{y_i\bar{x_i}}$
$= \overline{\sum y_i\bar{x_i}}$
$= \overline{\sum x_i \bar{y_i}}$ ~~~~
③ $\sum x_i\bar{x_i} > 0$ as
$\Rightarrow \sum |x_i|^2 > 0$

$\therefore \sum x_i\bar{y_i}$ is an inner product

(iii) $V = C([0,1])$

$\langle f,g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt$

① trivial
② $\int_0^1 f(t)\overline{g(t)}\,dt$
$= \overline{\int_0^1 \overline{f(t)}g(t)\,dt}$
$\langle x,y\rangle = \overline{\langle y,x\rangle}$
③ $\int_0^1 |f(t)|^2\,dt > 0$

$\therefore \langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$ is a inner product

Defn : (1) A vectorspace $V$ with an inner product is called an inner product space

(11) Two vectors $x$ and $y$ in an inner product space are orthogonal if $\langle x,y\rangle = 0$

(111) A set of vectors $\{x_1,\dots,x_n\}$ in $V$ is orthonormal set of vectors if
Norm $\|x\|$ ← ① $\|x_i\| = \langle x_i,x_i\rangle^{1/2} = 1$
② $\langle x_i, x_j\rangle = 0$ $\forall$ $i\neq j$

Lemma : (Cauchy-Schwartz inequality)
In an inner product space $V$, $|\langle x,y\rangle| \leq \|x\|\|y\|$
where $\|x\| = \langle x,x\rangle^{\frac{1}{2}}$
Norm

proof: $\langle x+t\alpha y, x+t\alpha y\rangle \quad t\in\mathbb{R}$

$$= \langle x, x+t\alpha y\rangle + t\alpha\langle y, x+t\alpha y\rangle$$

$$= \langle x,x\rangle + \overline{\alpha t}\langle x,y\rangle$$

(assuming $x$ is not a scalar multiple of $y$) $+\ t\alpha\langle y,x\rangle + t\alpha\overline{t\alpha}\langle y,y\rangle$

$$= \|x\|^2 + \bar{\alpha} t\langle x,y\rangle + t\alpha\langle y,x\rangle + t^2|\alpha|^2\|y\|^2$$

$$= \|x\|^2 + t\left[\overline{\alpha\langle y,x\rangle} + \alpha\langle y,x\rangle\right] + t^2|\alpha|^2\|y\|^2$$

$$= \|x\|^2 + t\left[2\mathrm{Re}(\alpha\langle y,x\rangle)\right] + t^2|\alpha|^2\|y\|^2$$

$$= \|x\|^2 + 2t\,\mathrm{Re}(\alpha\langle y,x\rangle) + t^2|\alpha|^2\|y\|^2$$

choose $\alpha$ with $|\alpha|=1$ s.t $\alpha\langle y,x\rangle = |\langle y,x\rangle|$

then $\langle x+t\alpha y, x+t\alpha y\rangle = \|x\|^2 + 2t|\langle y,x\rangle| + t^2\|y\|^2 \geq 0 \quad \forall t\in\mathbb{R}$

$\Rightarrow \Delta < 0$

$\Rightarrow \cancel{4t^2}|\langle x,y\rangle|^2 < \cancel{4t^2}\|x\|^2\|y\|^2$

$\Rightarrow |\langle x,y\rangle| \leq \|x\|\,\|y\|$

lemma: (Triangle inequality) $\|x+y\| \leq \|x\| + \|y\|, \ \forall x,y\in V$

proof:

$$\|x+y\|^2 = \|x\|^2 + 2\mathrm{Re}\langle y,x\rangle + \|y\|^2 \quad (\text{putting } t\alpha = 1)$$

$$\leq \|x\|^2 + \|y\|^2 + 2|\langle y,x\rangle|$$

$$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\|$$

$$= (\|x\| + \|y\|)^2$$

Theorem: Let $V$ be a finite dimentional inner product space. If $\{y_1,\dots,y_k\}$ is a linearly independent set of vectors in $V$, then $\exists$ an orthonormal set of vectors $\{x_1,\dots,x_k\}$ s.t

(Gram Smitz decomposition) $x_i \in \mathrm{span}\{y_1,\dots,y_k\}$

(or $\exists$ a lin ind orthonormal set spanning $\{y_1,\dots,y_k\}$. If $\mathrm{span}\{y_1\dots y_k\}$ is $V$ then the new set is also a basis)

Proof: $\{y_1,\dots,y_k\}$ is a linearly ind set of vectors in $V$

wlog pick $y_1$

then

$$x_1 = \frac{y_1}{\|y_1\|} \quad (\text{so } \|x_1\| = 1)$$

now, $x_2$ s.t it is unit vector and $\langle x_1, x_2\rangle = 0$

now, $x_2 = c(y_2 - \langle y_2, x_1\rangle x_1)$

$c$: scalar

as $\langle x_2, x_1\rangle = c\langle y_2 - \langle y_2,x_1\rangle x_1, x_1\rangle$

$$= c\langle y_2, x_1\rangle - c\langle y_2,x_1\rangle\underbrace{\langle x_1,x_1\rangle}_{\|x_1\|^2}$$

$$= c\langle y_2, x_1\rangle - c\langle y_2, x_1\rangle$$

$$= 0$$

now $x_2 = c\left[ y_2 - \langle y_2, x_1 \rangle x_1 \right]$

now c for which $\langle x_2, x_2 \rangle = 1$

we can find c s.t $\|x_2\| = 1$

suppose that we have chosen $\{x_1, \ldots x_{n-1}\}$ s.t this set is orthonormal

then
$$x_n = \beta\left( y_n - \sum_{i=1}^{n-1} \langle y_n, x_i \rangle x_i \right)$$

now
$$\langle x_n, x_j \rangle = \beta \left\langle y_n - \sum_{i=1}^{n-1} \langle y_n, x_i \rangle x_i , x_j \right\rangle$$
$$= \beta\left[ \langle y_n, x_j \rangle - \langle y_n, x_j \rangle \right]$$
$$= 0$$

choose $\beta$ accordingly

**lemma**: An orthonormal set of vectors is linearly independent
$(\langle x_i, x_j \rangle = 0, \langle x_i, x_i \rangle = 1, \forall i \neq j)$

**proof**: If $\{x_1, \ldots x_n\}$ is an orthogonal set of vectors and if $c_1 x_1 + \cdots c_n x_n = 0$

then taking inner product with $x_i$

$$\langle c_1 x_1 + \cdots c_n x_n , x_i \rangle = \langle 0, x_i \rangle = 0$$
$$\left( \text{as } \langle 2 \cdot 0, x_i \rangle = \langle 0, x_i \rangle + \langle 0, x_i \rangle \right)$$
$$\langle 0, x_i \rangle$$

$\Rightarrow c_1 \langle x_1, x_i \rangle + \cdots c_n \langle x_n, x_i \rangle = 0$

$\Rightarrow c_i \langle x_i, x_i \rangle = 0$

$\Rightarrow c_i (1) = 0$

$\Rightarrow c_i = 0 \quad \forall i = 1, 2 \ldots n$

**Note**: any basis $\rightarrow$ orthonormal basis conversion

$y \in V$, $\langle \cdot, y \rangle : V \times V \longrightarrow \mathbb{C}$
$$x \longmapsto \langle x, y \rangle \in \mathbb{C}$$
$\therefore$ linear functional

**Theorm**: Let V be an inner product space. If f is a linear functional on V then $\exists$ a unique vector $y \in V$ s.t
$$f(x) = \langle x, y \rangle, \forall x \in V$$

**proof**: let $\{x_1, \ldots x_n\}$ be an orthonormal basis of inner product space V (so it is also a basis of V)

Suppose $f(x_i) = \alpha_i, \forall i = 1, 2, \ldots n$

set $y = \sum_{i=1} \bar{\alpha}_i x_i$

then $\langle x_i, y \rangle = \langle x_i, \sum \bar{\alpha}_i x_i \rangle$
$$= \overline{\bar{\alpha}_i} \langle x_i, x_i \rangle$$
$$\langle x_i, y \rangle = \alpha_i = f(x_i), \forall i = 1, \ldots n$$
$$\Rightarrow f(x) = \langle x, y \rangle, \forall x \in V$$

uniqueness of y: if $\exists y_2$ s.t
$$f(x) = \langle x, y_2 \rangle$$
$$\& \quad f(x) = \langle x, y \rangle$$

then $\langle x, y \rangle = \langle x, y_2 \rangle \ \forall x \in V$

$\Rightarrow \langle x, y - y_2 \rangle = 0 \ \forall x \in V$

$\Rightarrow \langle y - y_2, y - y_2 \rangle = 0$

$\Rightarrow y - y_2 = 0$

$\Rightarrow y = y_2$

Let $\mathcal{C}$ be a subspace of an inner product space $V$.

$$\mathcal{C}^{\perp} = \{ f \in V \mid f(y) = 0 \ \forall y \in \mathcal{C} \}$$

$$= \langle z \in V \mid \langle y, z \rangle = 0 \ \forall y \in \mathcal{C} \rangle$$

$\therefore \mathcal{C}^{\perp} =$ orthocondiment of $\mathcal{C}$

30th oct :

Recap :

$$J_n(\lambda) = \begin{bmatrix} \lambda & & 0 \\ 1 & \ddots & \\ 0 & 1 & \lambda \end{bmatrix}_{n\times n}$$ nilpotency index

$$J_n(\lambda) = \begin{bmatrix} \lambda & & 0 \\ 1 & \ddots & \\ 0 & 1 & \lambda \end{bmatrix}_{n\times n}$$

$$T = \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & 1 & 0 \end{bmatrix}_{n\times n} \qquad T^n = 0, \quad T^{n-1} \neq 0$$

$$T^{n-1}x \neq 0$$
$$T^n x = 0$$

$$\{x, Tx, \dots, T^{n-1}x\}$$

ordered basis then matrix we get is :

Jordan form $\leftarrow \begin{bmatrix} \lambda & & 0 \\ 1 & \ddots & \\ 0 & 1 & \lambda \end{bmatrix}$

Note : $\{T^{n-1}x, \dots x\}$ ordered basis then

$\begin{bmatrix} \lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{bmatrix}$ ← Transpose of jordan matrix

$V - \{x_1 \dots x_n\}$ ← orthonormal basis (ONB) for V

for any $x \in V$,

$$x = \sum_{i=1}^{n} c_i x_i \qquad \left( x = \sum \alpha_i x_i \text{, but } x_i \perp x_j \ \forall i \neq j \text{ i.e } \langle x_i, x_j \rangle = 0 \right)$$

To find $c_i$ : $\langle x, x_i \rangle = c_i$

as $\langle x_j, x_i \rangle$ for $i \neq j$ $= 0$

for $i = j$ $= 1$

$$\Rightarrow x = \sum_{i=1}^{n} \langle x, x_i \rangle x_i \qquad \left( \langle x, x_i \rangle = c_i \ \forall i \right)$$

Note : $x = \sum_{i=1}^{n} \langle x, x_i \rangle x_i$

now, $\|x\|^2 = \langle x, x \rangle = \left\langle \sum_{i=1}^{n} \langle x, x_i \rangle x_i, \sum_{i=1}^{n} \langle x, x_i \rangle x_i \right\rangle$

$\left( \|x\|^2 = \langle x, x \rangle \right)$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \langle x, x_i \rangle \overline{\langle x, x_j \rangle} \langle x_i, x_j \rangle$$

$$= \sum_{i=1}^{n} \langle x, x_i \rangle \overline{\langle x, x_i \rangle} \langle x_i, x_i \rangle$$

$$\|x\|^2 = \sum_{i=1}^{n} |c_i|^2$$

or

$$\|x\| = \sum_{i=1}^{n} |\langle x, x_i \rangle|^2 = \sum_{i=1}^{n} |c_i|^2$$

Note : There is a bijection b/w n dimensional (inner space V) and $\mathbb{C}^n$

$\psi : V \longrightarrow \mathbb{C}^n$ as $\langle x, x_i \rangle \in \mathbb{C}$

$x \longmapsto (\langle x, x_1\rangle, \langle x, x_2\rangle, \ldots \langle x, x_n\rangle)$

This is ① well defined as $x = \sum \langle x, x_i \rangle x_i$

② one-one
③ onto
④ linear

$\longrightarrow V \cong \mathbb{C}^n$

$\left( \begin{array}{l} x = \sum c_i x_i \\ \psi(x) = (c_0, c_1 \ldots c_n) \end{array} \right)$

**Note**: This $\psi$ preserves the inner product:

then $\langle x, x \rangle = \sum |c_i|^2$
$= \langle (c_0, c_1, \ldots c_n), (c_0, \ldots c_n) \rangle$
$= \langle \psi(x), \psi(x) \rangle$

$\langle x, x \rangle = \langle \psi(x), \psi(x) \rangle \; \forall x \in V$

see down ← & $\langle x, y \rangle = \langle \psi(x), \psi(y) \rangle \; \forall x, y \in V$

Polarising identity:

$$\langle x, y \rangle = \frac{1}{4}\left( \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 \right)$$

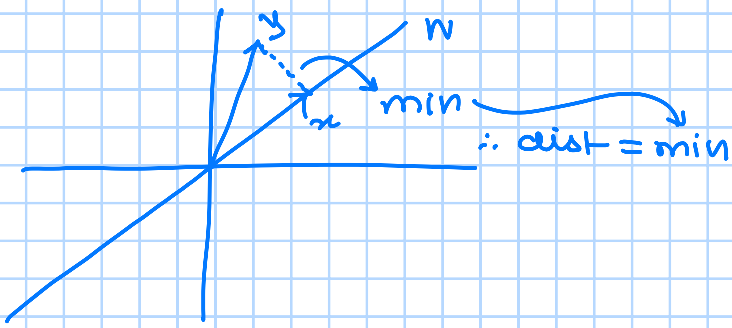
(using this we show $\langle x, y \rangle = \langle \psi(x), \psi(y) \rangle$)

**Note**: upto isomorphism both $V$ & $\mathbb{C}^n$ are same.

**Defn**: Let $W$ be a subspace of an inner product space $V$. A vector $x \in W$ is a best approximate of $y \in V$ if

$$\|y - z\| \geq \|y - x\| \;\; \forall z \in W$$

$$\text{dist}(W, y) = \text{Inf}\left\{ \|z - y\| ; \forall z \in W \right\}$$




note here $y - x \perp W$

**Theorm**: Let $W$ be a subspace of an inner product space $V$.

(a) Then $x \in W$ is the best approximation of vector $y$ iff $y - x \in W^\perp$

$\left( W^\perp = \{ v \in V \mid \langle v, w \rangle = 0, \forall w \in W \} \right)$
also called orthocompliment

(b) Best approximation point is unique

(c) If $\{w_1 \ldots w_r\}$ is an ONB for $W$ then the best approximation of $W$ of $y \in V$ is $\sum \langle y, w_i \rangle w_i$

Exe: $W^\perp$ is subspace of $V$.
as $W^\perp = \{ v \in V \mid \langle v, w \rangle = 0, \forall w \in W \}$ for $v_1, v_2 \in W^\perp$
$\langle \alpha v_1 + v_2, w \rangle = \alpha \langle v_1, w \rangle + \langle v_2, w \rangle = \alpha \cdot 0 + 0 = 0 \; \therefore \alpha v_1 + v_2 \in W^\perp$

proof :(b) supposing (a)

suppose $x_1, x_2 \in W$
& Both are best approx of $y$
then $y - x_1 \in W^\perp$
& $y - x_2 \in W^\perp$

$\Rightarrow$ $x_1 - x_2 \in W^\perp$
also
$x_1 - x_2 \in W$

i.e $\langle x_1 - x_2, x_1 - x_2 \rangle = 0$

$\Rightarrow x_1 - x_2 = 0$
$\Rightarrow x_1 = x_2$

(c) $y - \sum_{i=1}^{r} \langle y, w_i \rangle w_i \in W^\perp$ then we are done

$\langle y - \sum \langle y, w_i \rangle w_i, w_j \rangle$

$= \langle y, w_j \rangle - \langle y, w_j \rangle$
$= 0$
$\forall j = 1, 2 \dots r$

$\therefore$ $y - \sum \langle y, w_i \rangle w_i$ is ortogonal to all $w_j$
$\Rightarrow$ ortogonal to linear combination of $w_j$
$\Rightarrow$ orthogonal to $x$, $\forall x \in W$

$\forall x \in W$
$\therefore$ $\langle y - \sum \langle y, w_i \rangle w_i, x \rangle = 0$

$\Rightarrow y - \sum \langle y, w_i \rangle w_i \in W^\perp$

$\Rightarrow \sum \langle y, w_i \rangle w_i$ is best approximation by part (a)

(Here this means that
$\langle y - \sum \langle y, w_i \rangle w_i, w_j \rangle = 0$
$\forall j$; $\forall w \in W$
$\therefore \langle y - \sum \langle y, w_i \rangle w_i, w \rangle = 0$
$\Rightarrow y - \sum \langle y, w_i \rangle w_i \in W^\perp$
$\Rightarrow \sum \langle y, w_i \rangle w_i$ is the Best app. of $y$)

(a) $\|y - z\| \geq \|y - x\|$ $\forall z \in W$

$\|y - z\|^2 = \|(y - x) + (x - z)\|^2$

$= \langle (y - x) + (x - z), (y - x) + (x - z) \rangle$

$= \|y - x\|^2 + \|x - z\|^2 + 2Re\langle x - z, y - x \rangle$
$\geq \|y - x\|^2$

$\Leftrightarrow$

$\|x - z\|^2 + 2Re\langle y - x, x - z \rangle \geq 0$
$\forall z \in W$

$x - z \in W$ ← gives all vector
$\forall w \in W$,
$\langle y - x, x - z \rangle = \langle y - x, w \rangle$
for $x - z = w$
as $\forall z \in W$
$w$ also spans $W$

$\Leftrightarrow \|\alpha\|^2 + 2\mathrm{Re}\langle y-x, \alpha\rangle \geq 0 \quad \forall \alpha \in W$

this is only possible for $\forall \alpha \in W$
$\langle y-x, \alpha\rangle = 0$
$\Rightarrow y - x \in W^\perp$

($x$ is Best app iff $\|\alpha\|^2 + 2\mathrm{Re}\langle y-x, \alpha\rangle \geq 0 \quad \forall \alpha \in W$)

so if $2\mathrm{Re}\langle y-x, \alpha\rangle$ is non-zero then can we come up with $\alpha$ s.t $\|\alpha\|^2 + 2\mathrm{Re}\langle y-x, \alpha\rangle$ is less then zero

($\|\alpha\|^2 + 2\mathrm{Re}\langle y-x, \alpha\rangle \geq 0$ iff $\langle y-x, \alpha\rangle = 0$)

$\cancel{2}\mathrm{Re}\langle y-x, \alpha\rangle = -\cancel{2}\|\alpha\|^2$

$$\alpha = -\frac{\langle (y-x), \alpha\rangle \alpha}{\|\alpha\|^2}$$

($\alpha = -\frac{\langle y-x, \alpha\rangle \alpha}{\|\alpha\|^2}$ is counter example)

so $\|\alpha\|^2 - 2\mathrm{Re}(\,) = \frac{|\langle y-x, \alpha\rangle|^2 \|\alpha\|^2}{\|\alpha\|^4}$

$$-2\mathrm{Re}\left\langle y-x, \frac{\langle y-x, \alpha\rangle \alpha}{\|\alpha\|^2}\right\rangle$$

$$= \frac{|\langle y-x, \alpha\rangle|^2}{\|\alpha\|^2} - 2\mathrm{Re}\frac{\overline{\langle y-x, \alpha\rangle}\langle y-x, \alpha\rangle}{\|\alpha\|^2}$$

$$= \frac{|\langle y-x, \alpha\rangle|^2}{\|\alpha\|^2} - \frac{2|\langle y-x, \alpha\rangle|^2}{\|\alpha\|^2}$$

$< 0$

$\therefore y - x \in W$

Exe: polorising identity

$\|x+y\|^2 = \langle x+y, x+y\rangle$
$= \langle x, x+y\rangle + \langle y, x+y\rangle$
$= \langle x+y, x\rangle + \langle x+y, y\rangle$
$= \langle y, x\rangle + \langle x, y\rangle + \langle x, x\rangle + \langle y, y\rangle$

$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\langle x, y\rangle$

sim

$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\mathrm{Re}\langle x, y\rangle$

$4\mathrm{Re}(\langle x, y\rangle) = \|x+y\|^2 - \|x-y\|^2$

and

$\mathrm{Re}\langle x, y\rangle + i\,\mathrm{im}\langle x, y\rangle = \langle x, y\rangle$

where $\mathrm{im}\langle x, y\rangle = \mathrm{Re}(-i\langle x, y\rangle)$

$\mathrm{im}\langle x, y\rangle = \mathrm{Re}\langle x, iy\rangle$

note $4\mathrm{Re}\langle x, iy\rangle = \|x+iy\|^2 - \|x-iy\|^2$

then $\quad Re\langle x, iy\rangle = \frac{1}{4}\left[\|x+iy\|^2 - \|x-iy\|^2\right] = im\langle x, y\rangle$

$$Re\langle x, y\rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right]$$

$$\Rightarrow \langle x, y\rangle = \frac{1}{4}\left[i^0\|x+i^0y\|^2 + i^1\|x+i^1y\|^2 + i^2\|x+i^2y\|^2 + i^3\|x+i^3y\|^2\right]$$

$$\langle x, y\rangle = \frac{1}{4}\sum_{n=0}^{3} i^n\|x+i^ny\|^2$$

Exe: $\langle x, y\rangle = \langle \psi(x), \psi(y)\rangle$ using polarising identity

$x = \sum a_i w_i$ where $w_i \perp w_j \ \forall \ i \neq j$
$y = \sum b_i w_i$

now $\psi(x) = (a_0, a_1, \ldots\ldots a_n)$
$\psi(y) = (b_0, \ldots\ldots\ldots b_n)$

$$\langle x, y\rangle = \frac{1}{4}\sum_{c=0}^{3} i^c\|x+i^cy\|^2$$

now $\|\psi(x)\|^2 = \sum a_i^2$

$$\|\psi(x) + i^c\psi(y)\|^2 = \sum(a_i + i^c b_i)^2$$
$$= \sum\left(a_i^2 + (i^c)^2 b_i^2 + 2i^c a_i b_i\right)$$
$$= \sum a_i^2 + (i^c)^2\sum b_i^2 + 2i^c\sum a_i b_i$$
$$= \langle x, x\rangle + (i^c)^2\langle y, y\rangle + 2i^c\langle x, y\rangle$$

now $\frac{1}{4}\Big[\cancel{\langle x, x\rangle} + (1)\cancel{\langle y, y\rangle} + 2\langle x, y\rangle$
$+(1)(\cancel{\langle x, x\rangle} + (-1)\cancel{\langle y, y\rangle} - 2\cancel{\langle x, y\rangle})$
$+(-1)(\cancel{\langle x, x\rangle} + (1)\cancel{\langle y, y\rangle} + 2\langle x, y\rangle)$
$+(1)(\cancel{\langle x, x\rangle} + (-1)\cancel{\langle y, y\rangle} - 2\cancel{\langle x, y\rangle})\Big]$

$= \langle \psi(x), \psi(y)\rangle$

$= \langle x, y\rangle$ (after reducing)

$\Rightarrow \langle x, y\rangle = \langle \psi(x), \psi(y)\rangle$

4th Nov :

**Recall** : $W \subseteq V$ - inner product space
$\beta \in W$ is best app. to $\alpha \in V$ by vector in $W \Leftrightarrow \alpha - \beta \in W^\perp$

**Note** : $\{a_1 \dots a_n\}$ is ONB for $W$, then $\beta = \sum \langle \alpha, \alpha_i \rangle \alpha_i$

Best app $\quad \alpha \in V$

$\beta$ is unique

Now, $W \oplus W^\perp = V \left( \alpha = \underset{\in W}{\beta} + \underset{\in W^\perp}{(\alpha - \beta)} \right)$ as $\beta$ is unique

($W \oplus W^\perp$ is when $\forall \alpha \in V$ we have $\beta$ (unique) s.t $\alpha = \underset{\in W}{\beta} + \underset{\in W^\perp}{(\alpha-\beta)}$)

if ONB of $W = \{x_1 \dots x_n\}$
ONB of $W^\perp = \{y_1 \dots y_m\}$
then
ONB $W \cup$ ONB $W^\perp = \{x_1 \dots x_n, y_1 \dots y_m\}$
$= $ ONB for $V$

**Note** : $\forall x \in V \Rightarrow x \in W \oplus W^\perp$
or $x = \sum_{i=1}^{n} \langle x, x_i \rangle x_i + \sum_{j=1}^{m} \langle x, y_j \rangle y_j$

let $E_W : V \longrightarrow V$
s.t $E_W(x) = E_W\left( \sum \langle x, x_i \rangle x_i + \sum \langle x, y_j \rangle y_j \right)$
$= \sum_{i=1}^{n} \langle x, x_i \rangle x_i \leftarrow$ This is also the best approximation to $x$ by vectors in $W$

**Exe** : $E_W^2 = E_W$ or $E_W$ is a projection
$E_W(E_W(x)) = E_W(x)$ as $E_W(x) \in W$ & $E_W(w) = w$

**Exe** : Ran $E_W = W$ & ker $E_W = W^\perp$
Ran $E_W = W$ as $\forall w \in W$ $E_W(w) = w$ & ker $E_W = W^\perp$ (trivial)

**Note** : There is a bijective correspondence b/w subspaces of $V$ and orthogonal projections (This bijection only exist in orthogonal projections)
$E^2 = E$ (projection)

**Defn** : orthogonal projection on an inner product space is a projection $E$ with nullspace of $E$ :
$$\text{null } E = (\text{Ran } E)^\perp$$

**Example** : $E_W : V \longrightarrow V$ $\quad$ orthogonal proj (Null $E_W = (\text{Ran } E_W)^\perp$)
then Ran $E_W = W$
null $E_W = W^\perp = (\text{Ran } E_W)^\perp$

**Note** : If $\ell \in V'$ (or $\alpha(V, R)$) ($\ell \in \{T : V \to R$, $T$ is linear$\}$) or $\ell : V \to R$ ↳ linear
then
$\exists y \in V$ s.t $\ell(x) = \langle x, y \rangle$ $\forall x \in V$

$(\ell(x) = \langle x, y \rangle \in R)$

$T : X \to Y$
$T' : Y' \to X'$
$T'(\theta)(x) = \theta(Tx)$
$\forall \theta \in Y'$
$x \in X$
Transpose

$X \xrightarrow{T} Y \xrightarrow{\theta} F$

$T' : \theta \longmapsto \theta \circ T$

$T : X \to Y$
$T' : Y' \to X'$
$T'(\theta)(x) = \theta \circ T(x)$
$\in Y'$
as $\theta \circ T(x)$ is
$\theta : Y \to R \quad T : X \to R$
$\theta \circ (Tx) : X \to R$

**Theorm** : Let $V$ be an inner product space and $T$ be linear map on $V$ then there is a unique linear map (adjoint of $T$)
$T^* : V \to V$
s.t $\langle Tx, y \rangle = \langle x, T^* y \rangle$ $\forall x, y \in V$

$(\langle Tx, y \rangle = \langle x, T^* y \rangle)$ unique proof :

$x \mapsto \langle Tx, y \rangle$ (This is a linear functional on $V$ for fixed $y \in V$)

Then By Riesz representation theorem

$f : x \mapsto \langle Tx, y \rangle$

$\exists$ a unique $z \in V$ s.t

$f(x) = \langle x, z \rangle = \langle Tx, y \rangle \ \forall x \in V$

set $z = T^* y$ then

we have to check that

$T^* : V \to V$ $\therefore$ well defined

$y \mapsto z$

$T^*$ is linear map : for $\alpha_1 y_1 + \alpha_2 y_2 \in V$

$\langle x, T^*(\alpha_1 y_1 + \alpha_2 y_2) \rangle = \langle Tx, \alpha_1 y_1 + \alpha_2 y_2 \rangle = \bar{\alpha_1} \langle Tx, y_1 \rangle + \bar{\alpha_2} \langle Tx, y_2 \rangle$

$= \langle x, \alpha_1 T^*(y_1) \rangle + \langle x, \alpha_2 T^*(y_2) \rangle$

$\langle x, T^*(\alpha_1 y_1 + \alpha_2 y_2) \rangle = \langle x, \alpha_1 T^*(y_1) + \alpha_2 T^*(y_2) \rangle$

$\therefore T^*$ is linear

Also see that $T^* \approx T'$ for the case of $T'(\theta)(x) = \theta(Tx)$

here $y \in V \longleftrightarrow \theta_y(x) = \langle x, y \rangle$ ← fixed

$\theta_y(Tx) = \langle Tx, y \rangle$

$\theta_{T^*y}(x) = \langle x, T^* y \rangle$

as $T'(\theta)(x) = \theta(Tx)$

$\theta_{T^*y}(x) = \theta_y(Tx)$

as $\theta_y(Tx) = (\theta_y \circ T)(x)$

$V \xrightarrow{T} V \xrightarrow{\theta_y} F$

$\theta \mapsto \theta \circ T$

$\theta_{T^*y}(x) = (\theta_y \circ T)(x)$

Note : $T : V \to V$

$\{x_1, \ldots, x_n\} \to$ ONB for $V$

$T(x_i) = \sum_j \langle T(x_i), x_j \rangle x_j$

$A_{ji} = \langle T(x_i), x_j \rangle$

$A$ ↑ matrix Rep $= \begin{bmatrix} | & | & & | \\ T(x_1) & T(x_2) & \cdots & T(x_n) \\ | & | & & | \end{bmatrix}_{n \times n}$

as $\begin{array}{c|c} & T(x_1) \ T(x_2) \cdots T(x_n) \\ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} & \begin{bmatrix} \ & \ \\ \ & \ \end{bmatrix}_{n \times n} \end{array}$

A looks like:

$$\begin{bmatrix} \langle T(x_1), x_1\rangle & \cdots & \langle T(x_n), x_1\rangle \\ \langle T(x_1), x_2\rangle & \cdots & \langle T(x_n), x_2\rangle \\ \vdots & & \vdots \\ \langle T(x_1), x_n\rangle & \cdots & \langle T(x_n), x_n\rangle \end{bmatrix}_{n\times n}$$

$$\left( \begin{array}{l} \text{as } T: V \to V \\ \quad x \mapsto T(x) \\ T(x_i) = \sum \langle T(x_i), x_j\rangle x_j \\ \quad A_{ij} = \langle T(x_i), x_j\rangle \end{array} \right)$$

**Propn:** Let $V$ be an inner product space on $T: V \to V$ be a linear map. Let $\{x_1, \dots, x_n\}$ be an ONB for $V$. Then the matrix rep of $T^*$ corresponding to $\{x_1, \dots, x_n\}$ is <u>conjugate</u> <u>transpose</u> of the matrix rep of $T$ w.r.t $\{x_1, \dots, x_n\}$.

**proof:**

$$\left( \begin{array}{l} T^*_{ji} = \langle T^*(x_i), x_j\rangle \\ \quad = \overline{\langle x_j, T^*(x_i)\rangle} \\ \quad = \overline{\langle T(x_j), x_i\rangle} \end{array} \right)$$

$$T^*_{ji} = \langle T^*(x_i), x_j\rangle$$

$$= \overline{\langle x_j, T^*(x_i)\rangle}$$

$$T^*_{ji} = \overline{\langle T(x_j), x_i\rangle} = \overline{T_{ij}}$$

conj. transpose: $T^*_{ij} = \langle T^*(x_j), x_i\rangle$

$$\Rightarrow T_{ij} = \langle T(x_j), x_i\rangle$$

$$\Rightarrow \overline{T_{ij}} = \overline{\langle T(x_j), x_i\rangle}$$

**Defn:** A linear map $T: V \to V$ is self-adjoint if $T^* = T$

6th Nov :

Exercise : (i) $(T_1+T_2)^* = T_1^* + T_2^*$ $\quad \langle (T_1+cT_2)\alpha | \beta\rangle = \langle \alpha | (T_1+cT_2)^*\beta\rangle$
(ii) $(T^*)^* = T$ $\quad \Rightarrow \langle \alpha | T_1^*\beta\rangle + \langle \alpha | \bar{c}T_2^*\beta\rangle = \langle \alpha | (T_1+cT_2)^*\beta\rangle$
(iii) $(\lambda T)^* = \bar{\lambda} T^*$

defn: A linear map $T$ on an inner operator is normal if $TT^* = T^*T$
(defn for normal)

eg : (i) unitary map on IPS (inner product space)
$T : V \to V$ is unitary if
$\|Tx\| = \|x\|$
$\Leftrightarrow \langle Tx, Ty\rangle = \langle x, y\rangle \quad \forall x, y \in V$
$\Leftrightarrow \langle x, T^* Ty\rangle = \langle x, y\rangle, \ \forall x, y \in V$
$\Leftrightarrow \langle x, (T^*T - I)y\rangle = 0 \quad \forall x, y \in V$
$\Leftrightarrow T^*T = I$
as $\forall x \ \langle x, y\rangle = 0$
$\Rightarrow y = 0$
(orthogonal to all $x$)
$\Leftrightarrow T^*T = I = TT^* \Leftrightarrow T^* = T^{-1}$

(Note : Normal $\Rightarrow \exists$ Basis s.t $P^{-1}TP = \Lambda$ → diag)

(unitary : $\|T\alpha\| = \|\alpha\|$
$\langle T\alpha | T\beta\rangle = \langle \alpha | \beta\rangle$
$T^*T = TT^* = I$)

$\therefore$ unitary $\Rightarrow$ Normal

$T = [c_1 \ \cdots \ c_n]$

$T^* = \begin{bmatrix} \bar{c_1} \\ \bar{c_2} \\ \vdots \\ \bar{c_n} \end{bmatrix}$ ← conjugate transpose (adjoint)

$\begin{bmatrix} \bar{c_1} \\ \vdots \\ \bar{c_n} \end{bmatrix} [c_1 \ c_2 \ \cdots \ c_n] = I$

$\begin{bmatrix} \bar{c_1} c_1 & \bar{c_1} c_2 & \cdots & \bar{c_1} c_n \\ \vdots & \vdots & & \end{bmatrix} = I$

$A_{ij} = \bar{c_i} c_j$
$[\delta_{ij}] = [a_{ij}]$
$\delta_{ij}$ is s.t $i=j \Rightarrow 1$
else 0

$U$ ← unitary, every column ← ONB

Note : $\bar{c_i} \cdot c_j = \langle c_j, c_i\rangle = \delta_{ij}$
$\Rightarrow$ if $i=j$ then $\langle c_i, c_j\rangle = 1$
else $\langle c_i, c_j\rangle = 0$

or $\{c_1, c_2 \ldots c_n\}$ ONB

Note : A matrix is unitary iff the column vectors of the matrix is ONB
$(U^*U = UU^* = I \Leftrightarrow B = \{c_1 \ldots c_n\}$ is ONB

Defn : We say two linear maps $T_1$ and $T_2$ on an IPS are unitary equivalent if $\exists$ a unitary map on $V$ s.t

$U^* T_1 U = T_2$
$U$ is unitary $\Rightarrow U^* = U^{-1}$
$U^{-1} T_1 U = T_2$ ← orthonormal basis instead of normal basis
orthonormal basis

unitary eq if
$U^* T_1 U = T_2$
→ ONB or with that transformation

**Lemma 1:** Let $T$ be a normal map on IPS $V$. If $x$ is an eigenvector of $T$ with eigenvalue $\lambda$ then $x$ is an eigenvector $T^*$ with eigenvalue $\bar{\lambda}$.

**proof:**

$\langle TT^*y, y\rangle = \langle T^*Ty, y\rangle \quad \forall y \in V$

$\Leftrightarrow \langle T^*y, T^*y\rangle = \langle Ty, Ty\rangle$

$\Leftrightarrow \|T^*y\| = \|Ty\|$

($Tx = \lambda x$ ← eigenvalue of $T$
then $\langle Tx | x\rangle = \langle \lambda x | x\rangle$
$= \langle x | T^*x\rangle$
$= \langle x | \bar{\lambda} x\rangle$
or $T^*x = \bar{\lambda} x$)

**Exe:** If $T$ is normal $\Rightarrow$ $T - \lambda I$ is normal, $\lambda \in \mathbb{C}$

① $\|T^*y\| = \|Ty\|$

(T is normal then $TT^* = T^*T$)

$\|T^*y\| = \|Ty\|$

② $\|(T-\lambda I)x\| = \|(T-\lambda I)^* x\| = \|(T^* - \bar{\lambda} I)x\|$

$0 = \|(T-\lambda I)x\| = \|(T-\lambda I)^* x\|$ $\quad ((T-\lambda I)^* = T^* - \bar{\lambda} I)$

as $T-\lambda I$ is normal $\quad = \|T^* - (\lambda I)^* x\|$

$(T-\lambda I)(T^*-\bar{\lambda} I)$ $\quad \|T - \lambda I\, x\| = \|T^* - \bar{\lambda} I\, x\|$

$= TT^* + \lambda\bar{\lambda} I - \lambda T^* - \bar{\lambda} T$ $\quad \Rightarrow (T^* - \bar{\lambda} I)x = 0$

or $\bar{\lambda}$ is eigenvalue of $T^*$ (eigenvector $x$)

$(T^*-\bar{\lambda} I)(T-\lambda I) = T^*T - \lambda T^* - \bar{\lambda} T + \bar{\lambda}\lambda I$ (then use ①)

**Lemma 2:** Let $T$ be a linear map on IPS $V$. If $W$ is an invariant subspace for $T$ then $W^\perp$ is invariant for $T^*$.

**proof:** $\quad (T: V \to V)$ $\quad T(W) \subseteq W$ then $T^*(W^\perp) \subseteq W^\perp$

$V = W \oplus W^\perp$

for $x \in W^\perp$, $\quad$ now as $\forall x \in W \Rightarrow T(x) \in W$

$\langle T^*x, y\rangle = \langle \underset{\in W^\perp}{x}, \underset{\in W}{Ty}\rangle \quad \forall y \in W$ $\quad$ now $\forall y \in W^\perp$ we have

$= 0$ since $Ty \in W$ $\quad \langle \underset{\in W}{Tx} | \underset{\in W^\perp}{y}\rangle = 0 \Rightarrow \langle \underset{\in W}{x} | T^*y\rangle = 0$

$\Rightarrow T^*x \in W^\perp$ $\quad \Rightarrow T^*y \in W^\perp$

$\therefore \forall x \in W^\perp \Rightarrow T^*x \in W^\perp$

**Theorem:** Let $T$ be a linear map on an IPS $V$. Then $\exists$ an ONB of $V$ s.t the matrix rep. of $T$ wrt ONB is upper triangular.

**Application:** $\quad T: V \to V$

Let $T \in M_{n\times n}(\mathbb{C})$ be a normal $\quad \exists$ ONB of $V$ s.t

$\exists$ unitary matrix $U$ s.t $\quad U^*TU$ = upper triangle

$A = U^*TU$ is upper triangular

$A^*A = (U^*TU)^*(U^*TU)$
$= U^*T^*U\, U^*TU$
$= U^*TT^*U$
$= (U^*TU)(U^*T^*U)$
$= AA^*$

if $T$ is normal, then $\exists\, U^*TU = A$ → upper triangle

$AA^* = A^*A \Rightarrow A$ is normal

$\therefore$ upper triangular also normal

**Exe:** $(U^*TU)^* = U^*T^*U$ or $(AB)^* = B^*A^*$

as $(AB)^* = B^*A^*$

$\langle ABx | y\rangle = \langle x | (AB)^*y\rangle$
$= \langle Bx | A^*y\rangle$
$= \langle x | B^*A^*y\rangle = \langle x | (AB)^*y\rangle$

$A = [A_{ij}]$ $\quad A_{ij} = 0 \quad \forall i > j$

upper triangular

$$= \begin{bmatrix} A_{11} & \cdots & \vdots \\ & \ddots & \vdots \\ 0 & & A_{nn} \end{bmatrix}_{n\times n}$$

then $Ae_1 = A_{11}e_1$

$A^*e_1 = \overline{A_{11}}e_1$

$$A^* = \begin{bmatrix} \overline{A_{11}} & 0 & \cdots & 0 \\ \overline{A_{12}} & & & 0 \\ \vdots & & & \vdots \\ \overline{A_{n1}} & & & \overline{A_{nn}} \end{bmatrix}_{n\times n}$$

A is normal and also upper triangular

$AA^* = A^*A$

But as $A^* e_1 = \bar{A}_{11} e_1 \Rightarrow A_{1j} = 0 \ \forall j > 1$

similarly

$$\left.\begin{array}{l} A e_2 = A_{22} e_2 \\ A^* e_2 = \bar{A}_{22} e_2 \end{array}\right\} \Rightarrow A_{2j} = 0 \ \forall j \neq 2$$

If we repeat this, we get

$$A = \begin{bmatrix} A_{11} & & & O \\ & A_{22} & & \\ & & \ddots & \\ O & & & A_{nn} \end{bmatrix}_{n \times n}$$

$\therefore$ A is diagonal

$$A = U^* T U$$

**Theorm:** (Spectral theorm for Normal matrices) If $T \in M_{n \times n}(\mathbb{C})$ is a normal matrix, then $\mathbb{C}^n$ has an ONB consisting of eigenvectors of T.

**proof:** If $\dim T = 1$ then the theorm holds true. Suppose that the theorm is also true for all inner product spaces with dim has less than dim V.

suppose that $\lambda$ is an eigenvalue of $T^*$ with eigenvector $x$.

Set $W = \text{span}\{x\} \subseteq V$

then W is an invariant subspace for $T^*$

then from prev lemma

$W^\perp$ is invariant under $(T^*)^* = T$

$$T_1 = T\big|_{W^\perp} : W^\perp \to W^\perp$$

since $\dim(W^\perp) < \dim(V)$

then By induction hypothesis,

$W^\perp$ has an orthonormal basis $\{x_1, \ldots, x_{n-1}\}$ (say)

s.t

$T_1$ is upper triagular wrt $\{x_1, \ldots, x_{n-1}\}$

Consider the orthonormal basis

$$\left\{x_1, \ldots, x_{n-1}, \frac{x}{\|x\|}\right\} \text{ for } V$$

$x/\|x\|$ is unit

as $V = W \oplus W^\perp$

$\rightarrow$ ONB of V = ONB W $\cup$ ONB $W^\perp$

**claim:** T is upper triagular wrt $\left\{x_1, \ldots, x_{n-1}, \frac{x}{\|x\|}\right\}$

$T_1(x_j) = T(x_j) = * x_1 + \cdots * x_{n-1} + 0 \cdot x$ $\quad \forall j = 1, \ldots, n-1$

upper triangular

so

$$\begin{bmatrix} \boxed{\phantom{T_1}} & \begin{matrix} * \\ \vdots \\ * \\ * \end{matrix} \\ 0 \cdots 0 & * \end{bmatrix} = \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix}_{n \times n}$$

T(x)

Already upper triagular

## Self adjoint matrix :

$$A^* = A$$

**Note** : self-adjoint matrices has only real eigenvalues.
Real self-adjoint matrices have real eigenvector.

**Exe** : Are the eigen values of a unitary matrix are uni-modular.

as $U^*U = I = U^*U$
also
$\|U\alpha\| = \|\alpha\|$
$\Rightarrow \|\lambda\alpha\| = \|\alpha\|$
$\Rightarrow |\lambda|\|\alpha\| = \|\alpha\|$
$\Rightarrow |\lambda| = 1$
$\therefore$ every eigenvalue of $U$ is unimodular

Also $T$ is normal then $\exists P$ s.t

$$P^* T P = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & \ddots & \lambda_n \end{bmatrix}$$

$P$ → ONB, $\lambda_n$ → eigvalues of $T$

now, $\bar{\lambda}$ is eigvalue of $T^*$

$$(P^* T^* P) = \begin{bmatrix} \ddots \end{bmatrix}$$

if $(P^* T^* P)^* = P^*(P^* T^*)^*$
$= P^*(T P)$
$$= \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$P^* T^* P = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}^* = \begin{bmatrix} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_n \end{bmatrix}$$

now $T = T^* \Rightarrow \lambda_1 = \lambda_2$
$\Rightarrow \lambda_i = \bar{\lambda}_i \ \forall i$
$\Rightarrow \lambda_i$ are real